MA Physics — January 2013

General Relativity

T. Hertog

p129-130

1 In a space of n dimensions define a tensor

$$C_{abcd} = R_{abcd} + \alpha(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) + \beta R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

where α and β are constants. Show that C_{abcd} has the same symmetries as R_{abcd}. The coefficients α and β are chosen to set $C^a{}_{bad} = 0$. Determine them. With this extra condition C_{abcd} is called the Weyl tensor. Show that it vanishes if $n = 3$. Setting $n = 4$, how many independent components do R_{ab} and C_{abcd} have? What does the Weyl tensor represent physically? Show that in vacuum

$$\nabla^a C_{abcd} = 0$$

p 261,3
270

2 (a) What is the minimum mass M_2 of a Schwarzschild black hole that results from the coalescence of two Kerr black holes of equal mass M_1 and opposite angular momentum a? If $|a| \approx GM_1$, what fraction of the original mass can be radiated away?

(b) Let E and L be the energy and angular momentum per unit mass of a neutral particle in free fall within the equatorial plane, i.e. on a timelike ($\sigma = 1$) or null ($\sigma = 0$) geodesic with $\theta = \pi/2$, of a Kerr black hole. Show that the particle's Boyer-Lindquist radial coordinate r satisfies an equation of the following form (in units where $c = G = 1$)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = 0,$$

where λ is an affine parameter and V is an effective potential.

(c) The effective potential V above can be written as

$$V(r) = -\sigma\frac{M}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\sigma - E^2)\left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3}(L - aE)^2$$

Discuss in qualitative terms the effect of the rotation of the black hole on the radius of the circular photon orbits both for co- and counterrotating photons.

3 Show that once a spacecraft crosses the horizon of a Schwarzschild black hole, it will reach the singularity in a proper time less than πM no matter how the engines are fired.

4 (a) By replacing the time coordinate t by one of the radial null coordinates

$$u = t + \frac{M}{\lambda}, \qquad v = t - \frac{M}{\lambda}$$

show that the singularity at $\lambda = 0$ of the Robinson-Bertotti (RB) metric

$$ds^2 = -\lambda^2 dt^2 + M^2 \left(\frac{d\lambda}{\lambda}\right)^2 + M^2 d\Omega^2 \tag{1}$$

is merely a coordinate singularity. Identify the Killing horizon associated with $\frac{\partial}{\partial t}$ and calculate its surface gravity κ.
(b) By introducing the new coordinates (U, V), defined by

$$u = \tan\left(\frac{U}{2}\right), \qquad v = -\cot\left(\frac{V}{2}\right)$$

obtain the maximal analytic extension of the RB metric and deduce its Penrose diagram.
(c) The RB metric (1) describes the geometry of a particular black hole in the region near the horizon. Which one?

5 (a) Consider the Robertson-Walker universe that best fits the current observations, with density parameters $\Omega_{R0} = 10^{-4}$, $\Omega_{M0} = 0.3$ and $\Omega_{\Lambda 0} = 0.7$. Sketch the behavior of the three Ω's as a function of the scale factor a, on a log scale from $a = 10^{-35}$ to $a = 10^{+35}$. Indicate the Planck time, nucleosynthesis and today.
(b) Sketch the evolution of the scale factor in this universe. Indicate the period during which large-scale structures such as galaxies form. What would the universe have been like if Λ had been significantly larger?