Calculus I, **24/10/2016**

Reeksnr.:
Naam:

1. (a) Toon aan dat

$$\sin^{-1}\frac{x-1}{x+1} = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$$

Geef ook aan voor welke reële x-waarden deze uitdrukking zin heeft.

(b) Bereken $y' = \frac{dy}{dx}$ in het punt (x_o, y_o) wanneer gegeven is dat

$$\sqrt{x^2+y^2} = c\tan^{-1}\frac{y}{x}$$

als $c = \dfrac{4\sqrt{2}}{\pi}$, $(x_o, y_o) = (1,1)$

(c) Geef de algemene oplossing van de differentiaalvergelijking

$$y'' - 6y' + 13y = 0$$

met beginvoorwaarden $y(0) = 0, y'(0) = 1$.

(2.5 ptn)

Antwoord:

(a) ***1.1pts : 0.3 for the domain + 0.8 for the calculation***

As shown in Ex. 9, p. 195

Consider the function $f(x) = \sin^{-1}\dfrac{x-1}{x+1} - 2\tan^{-1}\sqrt{x}$. The function is defined where $x \neq -1$ and $x \geq 0$, so the domain of $f(x)$ is $[0, +\infty)$. The function is continuous (a combination of continuous functions) and differentiable (a combination of differentiable functions).

The first derivative of f is:

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \left(\frac{x-1}{x+1}\right)' - 2\frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})' \\
&= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{x+1-x+1}{\left(x+1\right)^2} - 2\frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{x+1}{\sqrt{(x+1)^2-(x-1)^2}} \cdot \frac{x+1-x+1}{(x+1)^2} - \frac{1}{\sqrt{x}(1+x)} \\
&= \frac{2(x+1)}{\sqrt{4x}(x+1)^2} - \frac{1}{\sqrt{x}(1+x)} \\
&= 0
\end{aligned}$$

and thus f is a constant function. If we substitute $x = 0$, then $f(0) = -\frac{\pi}{2}$ and $f(x) = f(0) = -\frac{\pi}{2}$ for every x in the domain. Making a final substitution using the definition of f,

$$f(x) = -\frac{\pi}{2} \rightarrow \sin^{-1}\frac{x-1}{x+1} = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$$

This is valid if $x \in [0, +\infty)$, the interval where f is constant.

Common mistakes:

- From $\sqrt{x}$ we get that $x \geq 0$. In this case $\lim_{x\to\infty} \frac{x-1}{x+1} = 1$, so no conflict with arcsin $\frac{x-1}{x+1}$
- If you show that $f(x)$ is constant, then you have to prove that it is equal to $-\frac{\pi}{2}$
- Don't confuse the *domain* with the *range*!

(b) ***0.8pts : 0.5 for the differentiation + 0.3 for the calculation***

First we impose the constraint $x \neq 0$. Using implicit differentiation:

$$\frac{1}{2\sqrt{x^2+y^2}}\left(x^2+y^2\right)' = c\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{y}{x}\right)'$$
$$\frac{x+yy'}{\sqrt{x^2+y^2}} = c\frac{x^2}{x^2+y^2}\cdot\frac{y'x-y}{x^2}$$
$$x+yy' = c\frac{y'x-y}{\sqrt{x^2+y^2}}$$

In $(1,1)$ and if $c = \dfrac{4\sqrt{2}}{\pi}$:

$$1+y' = c\frac{y'-1}{\sqrt{2}}$$
$$y' = \frac{4+\pi}{4-\pi}$$

Common mistakes:

- Implicit differentiation: y is treated as a function of x. So, e.g, $\frac{d}{dx}(x^2+y^2) = 2x + 2y\frac{dy}{dx}$
- Differentiation of a fraction $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

(c) ***0.6pts : 0.3 to find the form of the solution and the correct roots + 0.3 to impose the initial conditions correctly***

We have to solve a second order differential equation of the form $ay''+by'+cy=0$, with $a=0, b=-6, c=13$.

The characteristic equation is $r^2-6r+13=0$ with a determinant $D=-16<0$. The solutions of the differential equations will thus have the form of $e^{(k+i\omega)t}$.

The roots of the characteristic equation are:

$$r_{1,2} = \frac{6\pm i\sqrt{16}}{2} = 3\pm 2i$$

The solutions of the differential equation are:

$$y_1 = e^{(3+2i)t}$$
$$y_2 = e^{(3-2i)t}$$

and the general form, expressed with $\sin, \cos$ (*Case III, p. 205*):

$$y = Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$$

The constants A, B can be calculated from the initial conditions. Since $y(0) = 0$, $A = 0$ and then we can easily calculate y':

$$y'(t) = 2Be^{3t}\cos(2t) + 3Be^{3t}\sin(2t)$$

Applying the second condition, $y'(0) = 1$, we get $2B = 1 \rightarrow B = \frac{1}{2}$.
Thus the solution is:

$$y(t) = \frac{1}{2}e^{3t}\sin 2t$$

Common mistakes:

- Be careful in your calculations: $D = 36 - 52 = -16$
- If you find complex roots, $r = a \pm ib$, the corresponding solutions in terms of $\sin, \cos$ **do not** have the imaginary unit i! (so no $\sin(ibt), \cos(ibt)$)
- The derivative of $Be^{3t}cos(2t)$ **is not** $-6Be^{3t}\sin(2t)$! Use the product rule correctly!

2. Gegeven de functie $f(x) = \left(3x^{7/5} - \frac{\pi}{x}\right)\cos\left(\frac{\pi}{2} - x\right)$.

   (a) Benoem het domein van de functie $f(x)$. Is $(f \circ f)(x)$ een even of oneven functie, of geen van beide (leg uit).

   (b) Bereken de limiet
   $$\lim_{x\to 0} f(x)$$
   en argumenteer hoe de functie f continu uitbreidbaar is op heel $\mathbb{R}$.

   (c) Bereken de vergelijking van de raaklijn aan de grafiek van f in $x = -\pi$.

(2.5 ptn)

Antwoord:

a) *1.2pts : 0.4 for the domain + 0.8 to characterize the function*

The function is defined everywhere except $x = 0$. First we can re-write f by substituting $\cos(\frac{\pi}{2} - x) = \sin x$.

There is **no need** to calculate $(f \circ f)(x)$! It is easy to verify that $f(x)$ is an even function, as:

- $\sin x$ is an odd function
- $3x^{7/5} - \frac{\pi}{x}$ is also an odd function

To determine if $(f \circ f)(x)$ is even or odd:

$$(f \circ f)(-x) = f(f(-x)) = f(f(x)) = (f \circ f)(x)$$

and thus the function $(f \circ f)$ is even.

Common mistakes:

- The function f is defined when $x \neq 0$! There is no restriction for $\cos(\frac{\pi}{2} - x) = \ sinx$
- The function $\sin x$ is **not** an even function!
- The product of two uneven functions is an even function!

b) *0.7pts : 0.4 for the correct value of the limit + 0.3 for the continuity of f*

Since f is an even function, we only need to calculate the limit $lim_{x\to 0^+} f(x)$

$$\begin{aligned}
lim_{x\to 0^+} f(x) &= lim_{x\to 0^+}\left(3x^{7/5} - \frac{\pi}{x}\right)\sin x \\
&= lim_{x\to 0^+}\frac{3x^{7/5} - \pi}{x}\sin x \\
&= lim_{x\to 0^+}\left[(3x^{7/5} - \pi)\frac{\sin x}{x}\right] \\
&= lim_{x\to 0^+}\left[(3x^{7/5} - \pi)\right] lim_{x\to 0^+}\left[\frac{\sin x}{x}\right] \\
&= -\pi \cdot 1 \\
&= -\pi
\end{aligned}$$

If we want f to be continuous, then $f(0) = -\pi$. So f is:

$$f(x) = \begin{cases} \left(3x^{7/5} - \frac{\pi}{x}\right)\cos\left(\frac{\pi}{2} - x\right), x \neq 0 \\ -\pi, x = 0 \end{cases}$$

Common mistakes:

- The limit is **not** equal to zero, as the form $\pm\infty \cdot 0$ is **not** defined!
- To make f continuous for every x, we must define $f(0) = lim_{x\to 0} f(x) = -\pi$

c) ***0.6pts : 0.3 for f' + 0.3 for the equation of the line***

We know that f is differentiable in $x = -\pi$ with:

$$f'(x) = \left(\frac{21}{5}x^{2/5} + \frac{\pi}{x^2}\right)\text{sin}x + \left(3x^{7/5} - \frac{\pi}{x}\right)\text{cos}x$$

and thus $f'(-\pi) = -1 + 3\pi^{7/5}$. The general form of the equation of a tangent line at (x_o, y_o) is:

$$y - y_o = f'(x_o)(x - x_o)$$

The value of f at $x = -\pi$ is $f(-\pi) = 0$, so the equation of the tangent line at $x = -\pi$:

$$y = \big(-1 + 3\pi^{7/5}\big)(x + \pi)$$

Common mistakes:

- $\text{cos}\frac{\pi}{2} = 0$ and nothing else! $(\neq \frac{\sqrt{2}}{2})$ etc.