During this exam, k denotes an algebraically closed field.

Question 1

Let $f : Y \to X$ be a morphism of affine k-varieties. What is $Z(\ker(\mathcal{O}(f)))$?

Question 2

(a) Let R be a ring that is a principal ideal domain but not a field. What is the Krull dimension of R? Explain your answer.

(b) Let R be a Noetherian ring of Krull dimension 0. Show that R has finitely many maximal ideals. (*Hint: what does spec*(R) *look like?*)

(c) Give an example of a Noetherian ring with Krull dimension at least one such that that R has finitely many prime ideals.

Question 3

Let $r, s \in \mathbb{N}_0$ and take $N = rs + r + s$. Consider the map

$$\psi : \mathbb{P}^r_k \times \mathbb{P}^s_k \to \mathbb{P}^N_k$$

that sends the couple

$$((a_0 : \cdots : a_r), (b_0 : \cdots : b_s)) \in \mathbb{P}^r_k \times \mathbb{P}^s_k$$

to the point

$$(a_0b_0 : a_0b_1 : \cdots : a_rb_{s-1} : a_rb_s) \in \mathbb{P}^N_k.$$

Show that ψ is well-defined and injective. Show that its image is a closed subvariety of $\mathbb{P}^N_k$. This projective variety is called the product of $\mathbb{P}^r_k$ and $\mathbb{P}^s_k$, and the map ψ is called the *Segre embedding*. (*Hint: try to determine the homogeneous ideal associated to the image of* ψ*. It is convenient to denote the homogeneous coordinates on* $\mathbb{P}^N_k$ *by* z_{ij} *with* $i \in \{0, \ldots, r\}$ *and* $j \in \{0, \ldots, s\}$*.*)

Question 4

Exercise 5.5.4 from the course notes.

(a) Let f be a rational function on $\mathbb{A}^n_k$, for some integer $n > 0$. Show that there exist polynomials p and q in $k[x_1, \ldots, x_n]$ with $q \neq 0$ such that

$$\mathrm{dom}(f) = \{x \in \mathbb{A}^n_k \mid q(x) \neq 0\}$$

and such that $f = p/q$ on $\mathrm{dom}(f)$. Thus the set of points where f is not defined is either empty or a hypersurface in $\mathbb{A}^n_k$.

(b) Consider the subvariety

$$X = \mathbb{A}_k^2 \setminus \{(0,0)\}$$

of $\mathbb{A}_k^n$. Prove that the map

$$\mathcal{O}(\mathbb{A}_k^2) \to \mathcal{O}(X) : f \mapsto f_{|X}$$

is an isomorphism. Conclude that X is not affine.