Calculus II, **03/09/2017**

Reeksnr.:
Naam:

1. Stel dat P een punt is op een vlakke kromme $\mathcal{K}$ waarvan de kromming $\kappa_P \neq 0$. De 'kromtecirkel' (osculating circle) is de cirkel met straal $R = 1/\kappa_P$ doorheen het punt P waarvan het centrum Q in de richting van de eenheidsnormaal $\hat{\mathbf{N}}$ ligt. Met andere woorden, het centrum wordt bepaald door: $\overrightarrow{OQ} = \mathbf{r}(t_0) + \dfrac{1}{\kappa_P}\hat{\mathbf{N}}$.
   Beschouw de kromme $\mathbf{r}(t) = \cosh(t)\mathbf{i} + t\mathbf{j}$ met $t \in \mathbb{R}$.

   a) Ga na dat de booglengteparameterisatie gegeven wordt door $\mathbf{r}(s) = \sqrt{1+s^2}\mathbf{i} + \ln(s + \sqrt{1+s^2})\mathbf{j}$. De booglengte wordt gemeten vanaf $t = 0$. *(0.5 ptn)*

   b) Bepaal een parametervergelijking $\mathbf{r}(u)$ voor de kromtecirkel voor $s = 2\sqrt{2}$. *(2 ptn)*

Antwoord:

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2. a) Bereken de integraal van $f(x,y,z) = e^{-x^2-z^2}$ over het gebied tussen twee cylinders $x^2+z^2=4$ en $x^2+z^2=9$ met $1 \le y \le 5$ en $z \le 0$. *($1\frac{1}{2}$ ptn)*

   b) Bereken de integraal $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\,dz\,dy\,dx.$
   Hint: Dit kan door een minimum aan rekenwerk door eerst te converteren naar cylindrische coördinaten. *(1 pt)*

Antwoord:

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(2,5) 1] $\vec{r}(t) = \cosh(t)\,\hat{i} + t\,\hat{j}, \quad t \in \mathbb{R}$

(0,5) (α) For the path: $s = \int_0^t \left|\frac{d\vec{r}(\tau)}{d\tau}\right| d\tau = \int_0^t \sqrt{\sinh^2(\tau)+1}\, d\tau \Rightarrow$

$$\Rightarrow s = \int_0^t \cosh(\tau)\, d\tau = \sinh t \quad (1) \Rightarrow$$

$$\Rightarrow t = \operatorname{Arcsinh}(s) \Rightarrow t = \ln\left(s + \sqrt{1+s^2}\right)$$

And $\cosh(t) = \sqrt{\sinh^2 t + 1} \underset{(1)}{=} \sqrt{s^2+1}$

0,5 ps.

(2) (β) $\vec{T}(s) = \frac{d\vec{r}(s)}{ds} = \frac{s}{\sqrt{1+s^2}}\,\hat{i} + \frac{\hat{j}}{\sqrt{1+s^2}}$ 0,5 ps

$$\frac{d\vec{T}(s)}{ds} = \frac{\hat{i}}{(1+s^2)^{3/2}} - \frac{s\,\hat{j}}{(1+s^2)^{3/2}}$$

$k_\rho = \frac{1}{\rho} = \left|\frac{d\vec{T}(s)}{ds}\right| = \frac{1}{(1+s^2)} \rightarrow \rho = (1+s^2) \overset{s=2\sqrt{2}}{=\!=\!=} 9$ 0,5 ps

the radious

For the coordinates of the centre of the circle:

$\vec{OQ} = \vec{r}(t_0) + \frac{1}{k_\rho}\hat{N}, \quad \hat{N} = \frac{1}{k_\rho}\frac{d\hat{T}}{ds} = \frac{1}{\sqrt{1+s^2}}\left(\hat{i} - s\,\hat{j}\right)$ 0,5 ps

$\Downarrow$

$$\vec{OQ} = \vec{r}(s_0) + \frac{1}{k_\rho}\hat{N} \overset{s_0 = 2\sqrt{2}}{=\!=\!=} \left(3\hat{i} + \ln(2\sqrt{2}+3)\hat{j}\right) + 9\left(\frac{\hat{i}}{3} - \frac{2\sqrt{2}}{3}\hat{j}\right)$$

$$\Rightarrow \vec{OQ} = 6\hat{i} + \left(\ln(2\sqrt{2}+3) - \frac{2\sqrt{2}}{3}\right)\hat{j}$$

Equation of circle: $(x-6)^2 + \left(y - \ln(2\sqrt{2}+3) + \frac{2\sqrt{2}}{3}\right)^2 = 81$ 0,5 ps

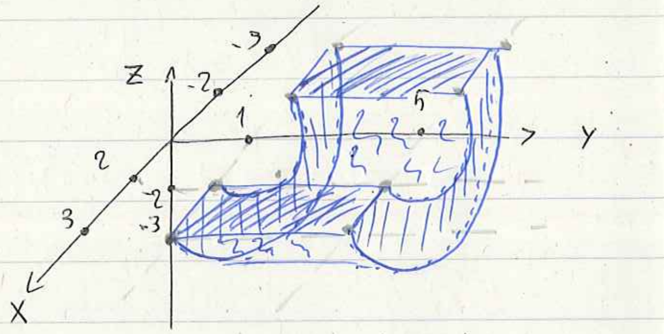
or

$$\left.\begin{cases} x = \rho\cdot\cos u + 6 \\ y = \rho\sin u + \left(\ln(2\sqrt{2}+3) + \frac{2\sqrt{2}}{3}\right) \end{cases}\right\} \rightarrow r(u) = x\,\hat{i} + y\,\hat{j}$$

2] (a) $f(x,y,z) = e^{-x^2-z^2}$

(1,5)

$$\left.\begin{array}{ll} x^2+z^2=4 & ① \\ x^2+z^2=9 & ② \\ 1 \le y \le 5 & \\ z \le 0 & \end{array}\right\} \Longrightarrow$$



· For $z=0 \Rightarrow x \in [2,3] \cup [-2,-3]$

For $x=0 \Rightarrow z \in [-2,-3]$

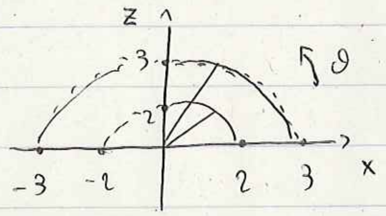
$y \in [1,5]$

· 0,5 ps for boundaries

We choose cylindrical coordinates (r,θ,y):

· $f = e^{-r^2}$, $① \Rightarrow r^2=4 \Rightarrow r=2$ ($r \geq 0$: radius)

$② \Rightarrow r^2=9 \Rightarrow r=3$

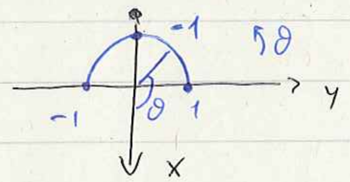


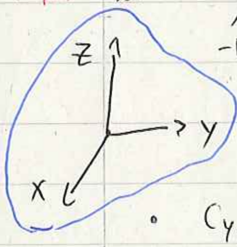
1p. { · 0,5 for calculations ; 0,5 for result }

$$\iiint f\, dV = \int_2^3 r e^{-r^2} \int_0^{\pi} d\theta \int_1^5 dy$$

$$= (5-1)\,\pi \left(-\frac{e^{-r^2}}{2}\right)\Bigg|_2^3 = 2\pi\left(e^{-4}-e^{-9}\right)$$

(1 p.) (b) $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y\, dz\, dy\, dx$ ①





· Cylindrical coordinates (r,θ,z):

· $18y = 18\, r\sin\theta$

$$\left\{\begin{array}{l} \sqrt{7-x^2-y^2} = \sqrt{7-r^2} \\ \sqrt{6(x^2+y^2)} = \sqrt{6}\, r^2 \\ \theta \in [+\pi/2, 3\pi/2] \\ r \in [0,1] \end{array}\right.$$

$\sqrt{6r^2} \le z \le \sqrt{7-r^2}$

0,5 ps for the boundaries

· For $x=0 \Rightarrow y = \pm 1$

For $x=-1 \Rightarrow y=0$

0,5 for the calculations + result

$$① \Rightarrow 18\int_0^1 r^2 dr \int_{+\pi/2}^{3\pi/2} \sin\theta\, d\theta \int_{\sqrt{6r^2}}^{\sqrt{7-r^2}} dz = 0$$

, because $\displaystyle\int_{+\pi/2}^{3\pi/2} \sin\theta\, d\theta = 0$ and the rest converge ($\in \mathbb{R}$ and $\neq \infty$)