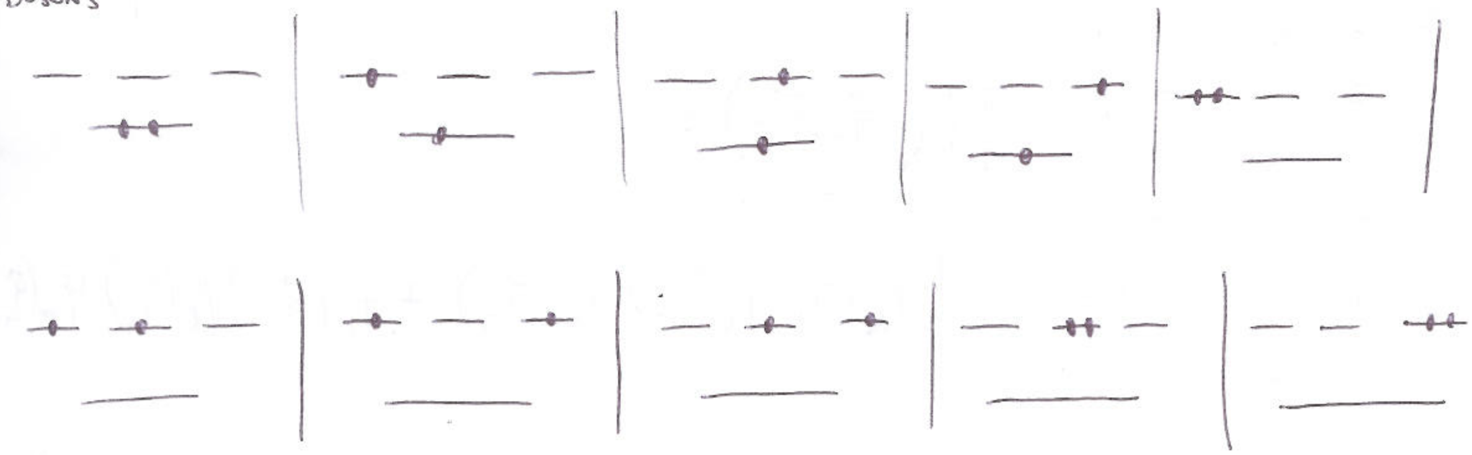
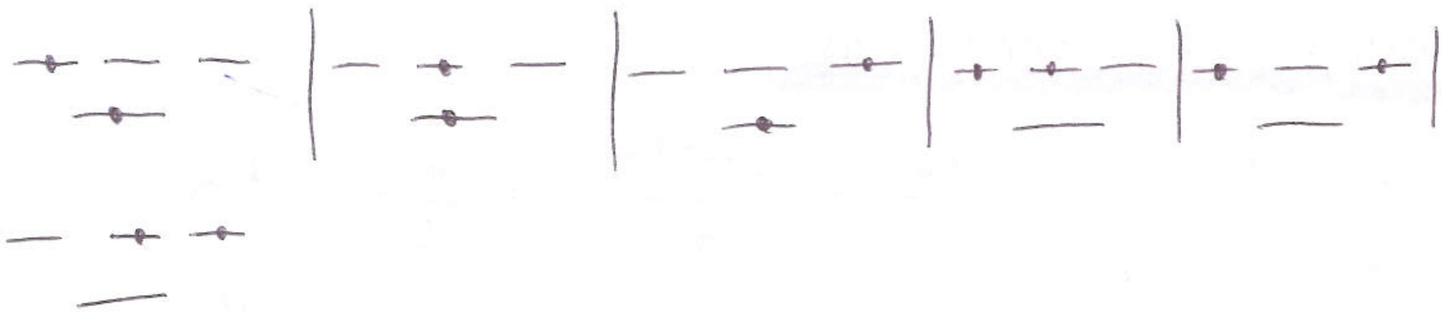
QUANTUM STATISTICS

Bosons



$$Z_{\text{Bosons}} = 1 + 3e^{-\beta\varepsilon} + 6e^{-2\beta\varepsilon}$$

Fermions



$$Z_{\text{Fermions}} = 3e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon}$$

LOW TEMPERATURE LIMIT FOR PRESSURE

Bosons : $\dfrac{pV}{k_BT} = \log \Xi = -\sum_\gamma \log\left[1-e^{-\beta(\varepsilon_\gamma-\mu)}\right]$

$\mu < \min_\gamma \varepsilon_\gamma$ hence low T $\dfrac{pV}{k_BT} \approx \sum_\gamma e^{-\beta(\varepsilon_\gamma-\mu)}$

$$\lim_{T\to 0} p = \lim_{T\to 0} \frac{k_BT}{V} \sum_\gamma e^{-\beta(\varepsilon_\gamma-\mu)} = 0$$

2

FERMION WAVEFUNCTIONS

k	l	m	sign
1	2	3	+
2	3	1	+
3	1	2	+
2	1	3	−
1	3	2	−
3	2	1	−

$$\Psi(\vec{q}_1, \vec{q}_2, \vec{q}_3) =$$

$$= \frac{1}{\sqrt{6}} \Big\{ \varphi_k(\vec{q}_1)\varphi_l(\vec{q}_2)\varphi_m(\vec{q}_3) + \varphi_k(\vec{q}_2)\varphi_l(\vec{q}_3)\varphi_m(\vec{q}_1)$$

$$+ \varphi_k(\vec{q}_3)\varphi_l(\vec{q}_1)\varphi_m(\vec{q}_2) - \varphi_k(\vec{q}_2)\varphi_l(\vec{q}_1)\varphi_m(\vec{q}_3)$$

$$- \varphi_k(\vec{q}_1)\varphi_l(\vec{q}_3)\varphi_m(\vec{q}_2) - \varphi_k(\vec{q}_3)\varphi_l(\vec{q}_2)\varphi_m(\vec{q}_1) \Big\}$$

GROUND STATE FERMI GAS

$$\vec{p} = \frac{h}{L}\vec{n} \qquad \frac{d\vec{p}}{(\Delta p)^3} = \frac{4\pi p^2 dp\, V}{h^3} = \frac{4\pi V}{h^3} p^3 \, d\log p$$

$$\varepsilon = \frac{p^2}{2m} \qquad d\log\varepsilon = 2\, d\log p \qquad = \frac{\cancel{4}^{2}\pi V}{h^3}(2m\varepsilon)^{3/2}\frac{1}{\cancel{2}}\frac{d\varepsilon}{\varepsilon}$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\sqrt{\varepsilon}\, d\varepsilon$$

$$g(\varepsilon) = V\alpha\sqrt{\varepsilon}$$

$$N = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = V\alpha \frac{2}{3}\varepsilon_F^{3/2} \qquad E_0 = \int_0^{\varepsilon_F} g(\varepsilon)\,\varepsilon\, d\varepsilon = V\alpha\frac{2}{5}\varepsilon_F^{5/2}$$

$$E_0 = V\alpha\frac{2}{5}\left(\frac{3N}{2V\alpha}\right)^{5/3} \qquad \begin{matrix} N \to \gamma N \\ V \to \gamma V \end{matrix} \qquad E_0 \to \gamma E_0$$

GAS OF MOLECULES

(a) $n\lambda_T^3 \ll 1$

(b) $$Z_{int} = \sum_{n=0}^{+\infty} (2n+1)\, e^{-\beta\alpha n} = \left(1 - \frac{2}{\alpha}\frac{\partial}{\partial\beta}\right) \sum_{n=0}^{+\infty} e^{-\beta\alpha n}$$

$$= \left(1 - \frac{2}{\alpha}\frac{\partial}{\partial\beta}\right) \frac{1}{1-e^{-\beta\alpha}} = \frac{1}{1-e^{-\beta\alpha}} + \frac{2\, e^{-\beta\alpha}}{(1-e^{-\beta\alpha})^2}$$

$$= \frac{1+e^{-\beta\alpha}}{(1-e^{-\beta\alpha})^2}$$

$$E_{int} = -\frac{\partial \log Z_{int}}{\partial\beta} = -\frac{\partial}{\partial\beta}\log\left(1+e^{-\beta\alpha}\right) + 2\frac{\partial}{\partial\beta}\log\left(1-e^{-\beta\alpha}\right) =$$

$$= \frac{\alpha e^{-\beta\alpha}}{1+e^{-\beta\alpha}} + 2\,\frac{\alpha e^{-\beta\alpha}}{1-e^{-\beta\alpha}} = \frac{\alpha}{e^{\beta\alpha}+1} + \frac{2\alpha}{e^{\beta\alpha}-1}$$

$$c_{int} = \frac{\partial E}{\partial T} = \frac{\partial\beta}{\partial T}\frac{\partial E}{\partial\beta} = -\frac{1}{k_B T^2}\frac{\partial}{\partial\beta}\left(\frac{\alpha}{e^{\beta\alpha}+1} + \frac{2\alpha}{e^{\beta\alpha}-1}\right)$$

$$= -\frac{1}{k_B T^2}\left\{\frac{-\alpha^2 e^{\beta\alpha}}{(e^{\beta\alpha}+1)^2} + \frac{-2\alpha^2 e^{\beta\alpha}}{(e^{\beta\alpha}-1)^2}\right\} =$$

$$= \frac{\alpha^2 e^{\beta\alpha}}{k_B T^2}\left[\frac{1}{(e^{\beta\alpha}+1)^2} + \frac{2}{(e^{\beta\alpha}-1)^2}\right]$$

High temperatures: $Z_{int} \approx \dfrac{2}{(\beta\alpha)^2}$ $\quad E_{int} = 2k_BT \quad c_{int} = 2k_B$

$$c_{TOT} = \underbrace{\frac{3}{2}Nk_BT}_{\text{translational degrees of freedom}} + 2Nk_BT = \frac{7}{2}Nk_BT$$

(4) BOSE-EINSTEIN CONDENSATION

$$\varepsilon = \alpha |\vec{p}|^s$$

(a) $\vec{p} = \frac{h}{L}\vec{n}$ $\quad \vec{n} = (n_x, n_y, n_z)$ $\quad n_{x,y,z} \in \mathbb{Z}$

$$\frac{d\vec{p}}{(\Delta p)^3} = \frac{4\pi V}{h^3} p^2 dp = \frac{4\pi V}{h^3} p^3 d(\log p)$$

$\varepsilon = \alpha p^s \qquad \log \varepsilon = \log \alpha + s \log p$

$$d(\log p) = \frac{1}{s} d(\log \varepsilon) = \frac{1}{s}\frac{d\varepsilon}{\varepsilon}$$

Hence: $\frac{4\pi V}{h^3} p^3 d(\log p) = \frac{4\pi V}{h^3}\left(\frac{\varepsilon}{\alpha}\right)^{\frac{3}{s}} \frac{d\varepsilon}{s\varepsilon}$

$$g(\varepsilon) = \frac{4\pi V}{s h^3} \frac{\varepsilon^{\frac{3}{s}-1}}{\alpha^{3/s}} = VA\,\varepsilon^{\frac{3}{s}-1} \qquad \text{with } A = \frac{4\pi}{s h^3 \alpha^{3/s}}$$

(b)
Particle number: $N = \int_0^{+\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}$

$\mu \to 0$ the integrand for small ε is $\approx \frac{\varepsilon^{\frac{3}{s}-1}}{\beta\varepsilon} \approx \varepsilon^{\frac{3}{s}-2}$

The integral converges if $\frac{3}{s} - 2 > -1$

$\frac{3}{s} > 1 \implies \boxed{s < 3}$ we have BEC.

This includes the case $s = 2$ $\quad \varepsilon = \alpha p^2$ ordinary free particles

(c)

$$\frac{pV}{k_B T} = \log \Xi = - \int_0^{+\infty} d\varepsilon \, g(\varepsilon) \log\left(1 - e^{-\beta(\varepsilon - \mu)}\right)$$

we define $z = e^{\beta\mu}$

If $s < 3$ we have BEC and we need in principle to consider the contribution from the ground state separately

Also for $T < T_{BEC}$ $z \simeq 1 - O\left(\frac{1}{V}\right)$

$$\frac{pV}{k_B T} = - \log(1 - z) - \int_0^{+\infty} d\varepsilon \, g(\varepsilon) \log\left(1 - e^{-\beta\varepsilon} z\right)$$

$$\frac{p}{k_B T} = O\left(\frac{\log V}{V}\right) - \int_0^{+\infty} d\varepsilon \, \frac{\cancel{V} \cdot A (\varepsilon)^{\frac{3}{s} - 1}}{\cancel{V}} \log\left(1 - e^{-\beta\varepsilon}\right)$$

here I put $z = 1$

$O\left(\frac{\log V}{V}\right) \to 0$ as $V \to \infty$

$$\frac{p}{k_B T} = - \int_0^{+\infty} d\varepsilon \, A \, \varepsilon^{\frac{3}{s} - 1} \log\left(1 - e^{-\beta\varepsilon}\right)$$ independent of V