Exam Differential Geometry January 2021

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Question 1 [4 points]

For all $a \in \mathbb{R}$, consider the vectorfield on $\mathbb{R}^3$ given by

$$X^a = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + a\frac{\partial}{\partial z}.$$

a

For all $a, b \in \mathbb{R}$, compute the lie bracket $[X^a, x^b]$.

b

For any point $p \in \mathbb{R}^3$, compute the integral curve τ_p^a of X^a satisfying $\tau_p^a(0) = p$.

c

For which values of $a \in \mathbb{R}$ and $p \in \mathbb{R}^3$ is τ_p^a a constant curve? (i.e. $\tau_p^a(t) = p \; \forall t$)

Question 2 [4 points]

Let N be a manifold of dimension n.

a

Let W be a compact, oriented manifold with (non-empty) boundary with $\dim(W) = n+1$. Endow its boundary with the orientation induced from W. Let $F : W \to N$ be a smooth map and denote by $f := D|_{\partial W} : \partial W \to N$ its restriction to the boundary of W. Show that for all $\omega \in \Omega^n(N)$ we have

$$\int_{\partial W} f^*\omega = 0.$$

b

Let M be a compact, connected and oriented manifold of dimension n. Consider two smooth maps $f_0, f_1 : M \to N$ which are smoothly homotopic. Show that for all $\omega \in \Omega^n(N)$ we have

$$\int_M f_0^*\omega = \int_M f_1^*\omega.$$

Remark: Smoothly homotopic means that there is a smooth map $F : [0,1] \times M \to N$ such that $F(0,p) = f_0(p)$ and $F(1,p) = f_1(p)$ $\forall p \in M$.

Question 3 [4 points]

Let $\mathrm{Mat}(2,\mathbb{R})$ be the Lie algebra of all real 2×2 matrices and let $\mathrm{GL}(2,\mathbb{R})$ be the Lie group of 2×2 invertible matrices.

a

Show that the exponential map exp:$\mathrm{Mat}(2,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{R})$ is not surjective.

b

Show that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not in the image of exp.
Hint: Relate the complex eigenvalues for any $A \in \mathrm{Mat}(2,\mathbb{R})$ to those of $\exp(A)$.

Question 4 [2 points]

Let $f : M \to N$ be a smooth map. Give a sufficient condition such that $f^{-1}(c)$ is a submanifold of N. Express its dimension in terms of $\dim(N)$ and $\dim(M)$.

Question 5 [2 points]

Let V be a four dimensional vector space. What is the dimension of $\bigwedge^2 V^*$? Given a basis of V, exhibit a basis of $\bigwedge^2 V^*$.

Question 6 [2 points]

Let M be an orientable and connected manifold of dimension n. Is it true that $H^n(M) \neq \emptyset$ always?

Question 7 [2 points]

Given a manifold M, there is a bijection between foliations and involutive distributions on M. Describe this bijection.