Calculus I, **11/01/2016**

Reeksnr.:
Naam:

1. Beschouw de reeks $\sum_{k=1}^{\infty} a_k$ gegeven door

$$\sum_{k=1}^{\infty} \frac{8k2^{k-1}}{10^k}.$$

(a) Toon aan dat

$$\lim_{k\to\infty} a_k = 0\,,$$

zodat alvast een nodige voorwaarde voor convergentie voldaan is.

(b) Toon op twee manieren aan dat de gegeven reeks convergeert, een keer met de verhoudingstest van D'Alembert, en een keer met het convergentiekenmerk (worteltest) van Cauchy.

(c) Toon aan dat de som van de reeks gelijk is aan $\frac{5}{4}$.
Hint: Gebruik het feit dat je kan schrijven dat

$$\sum_{k=1}^{\infty} k\left(\frac{1}{m}\right)^k = \sum_{k=1}^{\infty}\left(\frac{1}{m}\right)^k + \sum_{k=2}^{\infty}\left(\frac{1}{m}\right)^k + \sum_{k=3}^{\infty}\left(\frac{1}{m}\right)^k + \sum_{k=4}^{\infty}\left(\frac{1}{m}\right)^k + \dots$$

en gebruik eigenschappen van de geometrische reeks om de som uit te rekenen.

(2.5 ptn)

Antwoord:

a)

The $k-th$ term of the series is given by the formula $\alpha_k = \dfrac{8k\;2^{k-1}}{10^k}$ and can be expressed as:

$$\alpha_k = \frac{8k\;2^{k-1}}{10^k} = \frac{2^3k\;2^{k-1}}{10^k} = \frac{4k\;2^k}{10^k} = 4k\left(\frac{2}{10}\right)^k = 4k\left(\frac{1}{5}\right)^k$$

Then the limit of the $k-th$ term for $k\to\infty$ is:

$$\lim_{k\to\infty} a_k = 4\cdot\lim_{k\to\infty}\frac{k}{5^k} \overset{L'H\hat{o}pital}{=\!=\!=} 4\cdot\lim_{k\to\infty}\frac{1}{5^k\ln 5} = 0$$

which means that the necessary criterion for the series to be convergent is met.

b)

The D' Alembert ratio criterion:

First observe than $\left|\dfrac{\alpha_{k+1}}{\alpha_k}\right| = \dfrac{\alpha_{k+1}}{\alpha_k}$. Then the ratio can be written as:

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{\dfrac{8(k+1)\cdot 2^k}{10^{k+1}}}{\dfrac{8k\cdot 2^{k-1}}{10^k}} = \frac{10^k\cdot(8k\cdot 2^k + 8\cdot 2^k)}{8k\cdot 2^{k-1}10^{k+1}} = \frac{(8k\cdot 2^k + 8\cdot 2^k)}{10\cdot 8k\;2^{k-1}}$$

$$= \frac{2^k\cdot k + 2^k}{2^{k-1}\cdot 10k} = \frac{2^k\cdot(k+1)}{2^{k-1}\cdot 10k} = \frac{k+1}{k}\frac{2}{10} = \frac{k+1}{5k}$$

The limit of the ratio for $k \to \infty$ is:

$$\lim_{k\to\infty}\left|\frac{\alpha_{k+1}}{\alpha_k}\right| = \lim_{k\to\infty}\frac{k+1}{5k} = \frac{1}{5}\lim_{k\to\infty}\frac{k+1}{k} = \frac{1}{5} < 1$$

<u>The Cauchy root test:</u>

We observe again that $\left|\alpha_k\right| = \alpha_k$.We rewrite the sequence as:

$$\alpha^{1/k} = \left(\frac{8k\cdot 2^{k-1}}{10^k}\right)^{1/k} = \left(\frac{2^2\cdot k\cdot 2^k}{10^k}\right)^{1/k} = \left(\frac{4k}{5^k}\right)^{1/k} = \frac{1}{5}\left(4k\right)^{1/k}$$

The limit for $k \to \infty$ is:

$$\lim_{k\to\infty}\alpha_k^{1/k} = \frac{1}{5}\lim_{k\to\infty}\left(4k\right)^{1/k} = \frac{1}{5} < 1$$

From the two criteria we obtain that the series converges.

c)

We need to calculate the following sum:

$$\sum_{k=1}^{\infty}\alpha_k = \sum_{k=1}^{\infty}\frac{8k\ 2^{k-1}}{10^k} = \sum_{k=1}^{\infty}4k\left(\frac{1}{5}\right)^k$$

Using the hint, the sum can be expressed in the following form:

$$\begin{aligned}\sum_{k=1}^{\infty}4k\left(\frac{1}{5}\right)^k &= 4\sum_{k=1}^{\infty}k\left(\frac{1}{5}\right)^k\\ &= 4\left[\sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k + \sum_{k=2}^{\infty}\left(\frac{1}{5}\right)^k + \sum_{k=3}^{\infty}\left(\frac{1}{5}\right)^k + \sum_{k=4}^{\infty}\left(\frac{1}{5}\right)^k + ...\right]\end{aligned}$$

From this point different methods may be used. If you want to calculate the sums on the right side, you can derive the formula:

$$\sum_{k=m}^{\infty}r^k = \frac{r^m}{1-r}$$

Then we obtain:

$$\sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k = \frac{1/5}{1-1/5}$$

$$\sum_{k=2}^{\infty}\left(\frac{1}{5}\right)^k = \frac{\left(1/5\right)^2}{1-1/5}$$

$$\sum_{k=3}^{\infty}\left(\frac{1}{5}\right)^k = \frac{\left(1/5\right)^3}{1-1/5}$$

Thus the sum is:

$$\begin{aligned}\sum_{k=1}^{\infty}\alpha_k &= 4\left[\frac{1/5}{4/5}+\frac{\left(1/5\right)^2}{4/5}+\frac{\left(1/5\right)^3}{4/5}+\ldots\right]\\ &= \frac{4}{4/5}\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\ldots\right]\\ &= 5\sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k\\ &= 5\frac{1/5}{4/5}\\ &= \frac{5}{4}\end{aligned}$$

An alternative method is to factorise the right hand side of the equation given in the hint and work with partial sums.

Points

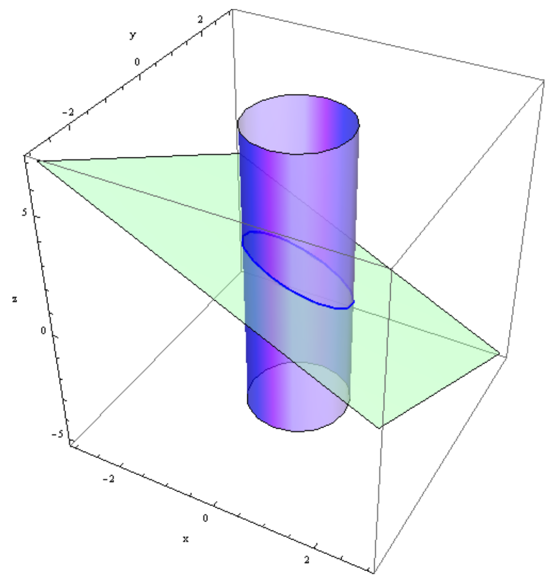
- 1a: 0.4p
- 1b: 0.4p for D'Alembert's criterion and 0.7p for Cauchy's root test.
- 1c: 1.0p

Although trivial limits do not require explanation, the limit of $k^{1/k}$ as $k \to \infty$ must be calculated (or at least an argument must be given)! It is also important (if you want to calculate a limit this way) to explain how does a form of $\infty \cdot 0$ behave as $k \to \infty$ (it is not trivial that it will be zero!).

2. De doorsnede van het vlak $x + y + z = 1$ met de cylinder $x^2 + y^2 = 1$ is een ellips. Bepaal de punten op die ellips die het dichtste bij en het verste van de oorsprong liggen. *(2.5 ptn)*

Antwoord:

The ellipse as the intersection of the plane $x + y + z = 1$ with the cylinder $x^2 + y^2 = 1$ is given in the following plot (not required to solve the exercise):



The distance of an arbitrary point P(x,y,z) from the origin is:

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

It is more convenient though to work with the square of the distance:

$$f(x, y, z) = x^2 + y^2 + z^2$$

We need to satisfy **two** constraints: the point belongs to the plane and to the cylinder, so we need to include two Lagrange multipliers in the Lagrangian:

$$\begin{aligned} L(x, y, z) &= f(x, y, z) + \lambda \cdot (x^2 + y^2 - 1) + \mu \cdot (x + y + z - 1) \\ &= x^2 + y^2 + z^2 + \lambda \cdot (x^2 + y^2 - 1) + \mu \cdot (x + y + z - 1) \end{aligned}$$

To find the critical points of L(x,y,z):

$$\frac{\partial L}{\partial x} = 0 = 2x + 2\lambda x + \mu \tag{1}$$

$$\frac{\partial L}{\partial y} = 0 = 2y + 2\lambda y + \mu \tag{2}$$

$$\frac{\partial L}{\partial z} = 0 = 2z + \mu \tag{3}$$

$$\frac{\partial L}{\partial \lambda} = 0 = x^2 + y^2 - 1 \tag{4}$$

$$\frac{\partial L}{\partial \mu} = 0 = x + y + z - 1 \tag{5}$$

From equations (1) and (2) we obtain:

$$(x-y)(\lambda+1)=0 \tag{6}$$

so either $x=y$ or $\lambda=-1$. We have to examine two cases:

Case I: $\lambda=-1$

From equation (1) we obtain:

$$2x-2x+\mu=0 \rightarrow \mu=0 \tag{7}$$

Then equation (3) is reduced to $z=0$, which means the points of interest in this case lie on the xy-plane.

Combining equations (4) and (5) we obtain that:

$$x^2+(1-x)^2=1 \rightarrow x=0 \quad or \quad x=1 \tag{8}$$

These values of x correspond to points A1(0,1,0) and A2(1,0,0). The distance from the origin equals to 1 for each point.

Case II: $x=y$

In this case equation (4) reduces to:

$$2x^2=1 \rightarrow x=\pm\frac{\sqrt{2}}{2} \tag{9}$$

We can calculate the corresponding value of z using equation (5). We get two points:
A3($-\frac{\sqrt{2}}{2}$,$-\frac{\sqrt{2}}{2}$,$1+\sqrt{2}$) and A4($\frac{\sqrt{2}}{2}$,$\frac{\sqrt{2}}{2}$,$1-\sqrt{2}$).

Out of these two points **only A3** is located at a maximum distance from the origin!

The requested points are:

- Minimum distance: A1, A2 with $d(A1)=d(A2)=1$
- Maximum distance: A3 with $d(A3)\simeq 6.82$

Points

- Correct form of Lagrangian: 0.8p
- Solve the equations for **both** cases: 1.2p
- Final answer regarding the points - only 1 maximum: 0.5p