General Relativity - Exercise session

Wednesday October 09 and Friday October 11, 2013

1. Let A be an event in spacetime with coordinates $(t_A, x_A, y_A, z_A) = (3, \sqrt{5}, 0, 0)$ in a given frame S.

   (a) Can you find a reference frame S', with the same origin as S, where the event A has coordinates $(t'_A, x'_A, y'_A, z'_A) = (2, 2, 1, 1)$?

   (b) Let now denote with B another event, with coordinates $(t_B, x_B, y_B, z_B) = (2, 0, 0, 0)$ in S. Is it possible to find another frame S'', still with the same origin as S, where $(t''_A, x''_A, y''_A, z''_A) = (t_B, x_B, y_B, z_B)$? What are the coordinates $(t''_B, x''_B, y''_B, z''_B)$ of event B in frame S''?

2. Frame S' moves with velocity $\vec{v}$ relative to frame S. A rod in frame S' makes an angle θ' with respect to the forward direction of motion. What is this angle θ as measured in S?

3. Prove the following statements.

   (a) If S^μ is timelike and $S^\mu P_\mu = 0$ then P_μ is spacelike.

   (b) Two ortogonal lightlike vectors are proportional.

4. [*Based on Carroll Ch.* 1 *Ex.* 6]
   In Euclidean three-space, let p be the point with coordinates $(x, y, z) = (1, 0, -1)$. Consider the following curves that pass through p:

$$x^i(\lambda) = (\lambda, (\lambda - 1)^2, -\lambda)$$
$$x^i(\mu) = (\cos\mu, \sin\mu, \mu - 1)$$

   (a) Calculate the components of the tangent vectors to these curves at p in the coordinate basis $\{\partial_x, \partial_y, \partial_z\}$.

   (b) Let $f = x^2 + y^2 - yz$. Calculate the expressions $df/d\lambda$ and $df/d\mu$ and compute their explicit values in p. What is the relation between $df/d\lambda$, df and the tangent vector to $x^i(\lambda)$?

5. For electric and magnetic fields, show that $B^2 - E^2$ and $\vec{E} \cdot \vec{B}$ are invariant under Lorentz transformations. Are there any invariants, quadratic in $\vec{B}$ and $\vec{E}$, that are not merely algebraic combination of two above?

6. Consider the infinite cylinder $S^1 \times \mathbb{R}$.

   (a) Describe it as an embedded surface in the three-dimensional space $ds^2 = dx^2 + dy^2 + dz^2$, *i.e.* describe the subspace $S^1 \times \mathbb{R}$ of $\mathbb{R}^3$ as an equation in (x, y, z). Parametrize this space using two coordinates and find the induced metric on this space.

(b) In contrast to the circle S^1, show that the infinite cylinder $S^1 \times \mathbb{R}$ can be covered with just one chart, by explicitly constructing the map.

7. Show that the determinant of the metric tensor $g = \det(g_{\mu\nu})$ is not a scalar.

8. Consider the following change of coordinates

$$\begin{aligned} t' &= t \\ x' &= x\cos\omega t + y \sin\omega t \\ y' &= -x\sin\omega t + y\cos\omega t \\ z' &= z \end{aligned}$$

where ω is a constant. Find the transformed components for the tangent vector with components $u^\mu(x) = \delta^\mu_t$.

9. [*Carroll Ch.* 2 *Ex.* 9]
In Minkowski space, suppose $*F = q\sin\theta d\theta \wedge d\phi$.

   (a) Evaluate $d(*F) = *J$

   (b) What is the two form F equal to?

   (c) What are the electric and the magnetic fields equal to for this solution?

10. Prove (3.33) and (3.34) in Carroll.
Hint: (3.33) immediately follows from $\partial_\lambda g = g g^{\rho\sigma}\partial_\lambda g_{\rho\sigma}$.

11. On the surface of a two-sphere, $ds^2 = d\theta^2 + \sin\theta^2 d\phi^2$, the vector $\mathbf{A}$ is equal to $\mathbf{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. What is $\mathbf{A}$ after is parallel transported around the circle $\theta = \theta_0$? What is the norm of $\mathbf{A}$?

12. Consider the spacetime metric $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ and the vector field $u^\alpha = e^\psi(\delta^\alpha_t + \Omega\delta^\alpha_\phi)$, with ψ and Ω functions of r and θ only.

   (a) Impose the normalization condition $u_\alpha u^\alpha = -1$ to compute e^ψ.

   (b) Find what are the conditions under which the for the vector field u^α is the four velocity of a geodesic trajectory, i.e. the tangent vector to a geodesic.

Notation: δ^α_β *is just the Kronecker's delta, so for instance* $u^t = e^\psi, u^r = 0, \ldots$ *in* u^α *above.*