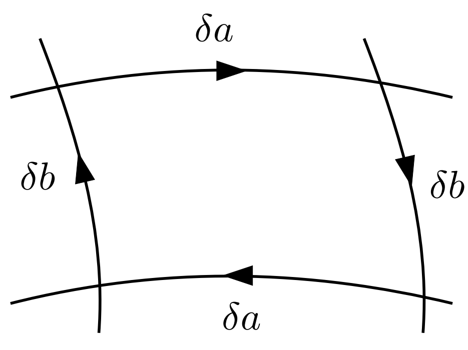
General Relativity - Exercise session

Friday October 25, 2013

1. Consider the metric space
$$ds^2 = dr^2 + r^2 d\theta^2$$
   (a) Write the two equations that results from the geodesic equation and show that the following are first integral of these:
$$r^2 \frac{d\theta}{ds} = R_0 = const$$
$$\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = 1\,.$$
   (b) Use this to get a first-order differential equation for $r(\theta)$, *i.e.* eliminate s as a parameter and replace it by θ.
   (c) Using the fact that the metric space is just flat 2-dimensional Euclidean space, write down the general equation for a straight line in r, θ coordinates and show that the straight line satisfies the equation in (b).

2. [*Carroll Ch.* 3 *Ex.* 7] or equivalently:
An infinitesimal circuit in the shape of the parallelogram can be specified by the differential displacements $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ representing the side of the parallelogram. Let a vector $\mathbf{A}$ be parallel transported around this circuit (*i.e.* displace it successively by $\delta\boldsymbol{a}, \delta\boldsymbol{b}, -\delta\boldsymbol{a}, -\delta\boldsymbol{b}$). Show that at the lowest non trivial order in $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ the change in $\mathbf{A}$ due to parallel transport around the circuit is
$$\delta A^\alpha \sim R^\alpha{}_{\beta\gamma\theta} A^\beta \delta a^\gamma \delta b^\theta$$



3. By examining the relative acceleration of a family of test particle trajectories in Newtonian gravity and comparing with the Newtonian limit of the equation of geodesic deviation (see Carroll § 3.10), derive the correspondence
$$R_{j0i0} = \frac{\partial^2 \Phi}{\partial x^i \partial x^j}$$

between the Newtonian potential and the Riemann tensor.
(A Newtonian test particle is acted on only by gravity; a test particle in general relativity follows a geodesics.)

4. Similarly to the gauge invariance of electromagnetism, general relativity has a symmetry of the equations of motion called diffeomorphism invariance:

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu,$$

where ξ^μ is an arbitrary vector. Use this to argue that, in two spacetime dimensions, we can write $g_{\mu\nu} = e^\phi \eta_{\mu\nu}$.

5. [*Carroll Ch. 7 Ex. 4*]
Show that the Lorenz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ is equivalent to the harmonic gauge condition. This gauge is defined by

$$\Box x^\mu = 0,$$

where each coordinate x^μ is thought of as a scalar function on spacetime.
[You might want to use $\Box\phi = (1/\sqrt{|g|})\partial_\mu(g^{\mu\nu}\sqrt{|g|}\partial_\nu)\phi$, in which case it is useful to know the expansion $\det(\eta_{\mu\nu} + h_{\mu\nu}) \simeq \det(\eta_{\mu\nu})(1 + h^\mu{}_\mu + \dots)$ valid for small metric perturbations $h_{\mu\nu}$ around flat Minkowski.]