General Relativity - Exercise session

Friday November 15, 2013

1. Let $T_{\mu\nu}$ be the energy-momentum tensor associated to an electro-magnetic field in vacuum:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu}{}^{\alpha}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) .$$

   (a) Show that the energy-momentum tensor is traceless and conserved on-shell.

   (b) Show that Einstein's equation sourced by this energy-momentum tensor reduces to

$$R_{\mu\nu} = \kappa T_{\mu\nu} .$$

2. Show that in linearised theory there is no attractive gravitational force between two thin parallel beams of light.
Hint: consider the photons of one of the two beams, with energy-momentum tensor taking the null-dust form, as the source of the gravitational field. In the linearised theory with such a source consider the other beam as a probe, i.e. consider a photon in geodesic motion in the gravitational field generated by the first beam.

3. Consider in Schwarzschild geometry the simplest case of radial motion and, assuming zero velocity at infinity ($u^{\mu} = \delta^{\mu}_{0}$), study the infall of a particle from any radius R to $r = 2M$.

   (a) How much proper time does it take? That is, how much time is elapsed on the particle's clock?

   (b) And how much coordinate time t elapses as the particle falls?
   Hint: Examine this in the case the particle is near $r = 2M$, i.e. for $r - 2M = \varepsilon \ll 1$

4. *[See e.g. Wald § 6.3]*
Consider Schwarzschild geometry and a photon coming from infinity with impact parameter b.

   (a) Show that if $b^2 < 27M^2$ the photon crosses the Schwarzschild radius.

   (b) Show that for b^2 suitably close to $27M^2$, the photon can be made to orbit an arbitrary number of times before escaping to infinity.