General Relativity - Exercise session

Friday November 29, 2013

1. Use Birkhoff's theorem to argue that a test particle experiences no gravitational forces inside a self-gravitational hollow sphere.

2. Consider Schwarzschild geometry and *outgoing Eddington-Finkelstein* coordinates obtained by the transformation

$$dt = du + \frac{dr}{1 - \frac{2M}{r}}.$$

   (a) What is the form of the Schwarzschild metric in these coordinates?

   (b) Now, let M be a function of the null coordinate u. Show that the spacetime is not vacuum, find the corresponding energy-momentum tensor and give a physical interpretation. [This is known as Vaidya geometry]

   (c) What if we were to consider *ingoing Eddington-Finkelstein* coordinates $dt = dv - \frac{dr}{1-\frac{2M}{r}}$ and M to be a function of v?

3. (a) Consider the Schwarzschild solution for a black hole of mass M. Ignoring the angular part for simplicity, find the *near horizon metric*, that is the metric as viewed parametrically close to the horizon and show that correspond to the *Rindler* spacetime

$$ds^2 = -\rho^2 d\sigma^2 + d\rho^2.$$

   (b) Show by a coordinate transformation that this metric is that of a constantly accelerating observer in Minkowski space with metric given by

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2),$$

   where a is the acceleration of the observer. In particular show that Rindller spacetime corresponds to the wedge $x > |t|$ of Minkowski spacetime.

   (c) Consider now the two-dimensional Milne universe

$$ds^2 = -d\tau^2 + \tau^2 d\chi^2,$$

   where $\tau > 0$ and χ is real. Is the singularity in $\tau = 0$ a true cosmological singularity? A way to answer this question is to show by explicit change of coordinates that in fact Milne spacetime corresponds to a wedge of two-dimensional Minkowski spacetime. However, to find the appropriate change of coordinates is no easy task in general. A more systematic way consists in studying null geodesics in this spacetime.
   [For this second approach see for instance Wald § 6.4. To summarize: Incompleteness of geodesics signals the presence of a true curvature singularity. A way to identify a coordinate singularity in two spacetime dimensions is to study null geodesics and to

use the affine parameters along such ingoing and outgoing geodesics as coordinates. In fact, the only coordinate singularities which can result from using null coordinates in two-dimensional spacetimes arise from bad parametrisation of geodesics. This can be investigated and corrected by comparing the coordinate parametrisation with an affine parametrisation.

4. Consider the Kerr metric with mass M and angular momentum a.

   (a) Show that the two zeros $r_+ > r_-$ of the function Δ are Killing horizons of the Killing vector fields

   $$\xi_\pm = \partial_t + \Omega_\pm \partial_\phi,$$

   where $\Omega_\pm$ are constants that you should determine. One interpretation of this result is that the event horizon (i.e., the outer Killing horizon $r = r_+$) of the Kerr black hole rotates with angular velocity Ω_+.

   (b) Show that the area of the event horizon of the Kerr black hole is

   $$A = 8\pi(M^2 + \sqrt{M^4 - J^2}).$$

   (c) One can prove that on the event horizon surface gravity is given by

   $$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}.$$

   Derive a condition for vanishing κ in terms of M and a.