Exam Statistical Mechanics

21 December 2021, 14:00-16:00

Classical Statistical Mechanics

Landau Theory

(4 points) In the Landau theory the free energy is expanded in powers of the order parameter m, as the magnetization per particle in a magnetic system. We consider the following Landau free energy per particle

$$\mathcal{F} = a(T - T_c)m^2 + bm^6 - hm \tag{1}$$

with $a > 0$ and $b > 0$ some parameters, h the magnetic field and T_c the critical temperatur. Find the critical exponents β and δ for this system.

Non-interacting spins

(4 points) Consider a system of N non-interacting spins in a magnetic field H. The spins can assume three states $s_i = -1, 0, +1$. The Hamiltonian of the systems is given by

$$\mathcal{H} = -H \sum_i s_i. \tag{2}$$

The total magnetization is defined by

$$M = \sum_i s_i. \tag{3}$$

Calculate the average magnetization $\langle M \rangle$ and the variance $\Delta M^2 \equiv \langle (M - \langle M \rangle)^2 \rangle$ and show that the relative fluctuations $\Delta M^2 / \langle M \rangle^2$ vanish in the thermodynamic limit $N \to \infty$. Calculate $\langle N_0 \rangle$, the average number of spins for which $s_i = 0$.

Quantum Statistical Mechanics

Three fermions

(4 points) Consider three identical non-interacting fermions in three different energy levels described by the wave functions $\phi_k(\vec{q})$, $\phi_l(\vec{q})$, and $\phi_m(\vec{q})$. Use this to determine the three-particle wave function $\Psi(\vec{q_1}, \vec{q_2}, \vec{q_3})$. Make sure to normalize your result properly.

1d anharmonic oscillator

(4 points) Consider a 1d *anharmonic* oscillator with energy levels $\epsilon_n = \hbar\omega(n + xn^2) + \epsilon_0$ where n is a positive integer and ϵ_0 is the ground state, or vacuum, energy. The system is chracterized by the frequency ω and the parameter x describing the deviation from the harmonic oscillator. What are the units of x? Assume that x is small and compute the first correction to the partition functionn of the harmonic oscillator. Consider the regime of large temperature, $\beta \ll \hbar\omega$, and compute the two leading contributions to the average energy and the specific heat c_V. Discuss your results paying particular attention to expectations from the equipartition theorem.

c_V at low T

(4 points) Consider a statistical system governed by quantum mechanics. Assume that the system is in equilibrium and at very low temperature T. Write down the partition function of the system in the canonical ensemble using an approximation in which only the first two states with energies ε_1 and $\varepsilon_2 = \varepsilon_1 + \delta\varepsilon$ contribute. Compute the average energy, E, in this approximation. Compute the specific heat, $c_V = \frac{\partial E}{\partial T}$ and discuss its behavior at low temperatures.