Exam Functional Analysis

KU Leuven, January 22, 2015

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for four hours. You are allowed to eat or drink.
- After one hour, you hand in your solution for Exercise 1. During the rest of the time, you work on the rest of the exercises and you will have your oral exam about Exercise 1.
- The exam is open book. This means that you may use the lecture notes and your own notes. You are not allowed to use anything else.
- This part of the exam counts for 12 of the 20 points. Every of the four exercises has the same weight.
- Write your name on every sheet that you hand in!

Good luck!

Exercise 1.

Let H be a Hilbert space, and let $B(H)$ be the space of bounded linear operators on H. In this exercise, we consider the strong topology and the weak topology on $B(H)$, as introduced in Example 7.4 of the lecture notes. Recall that the strong topology is defined by the family $\mathcal{P}_{\text{strong}}$ of seminorms given by

$$\mathcal{P}_{\text{strong}} = \{T \mapsto \|Tx\| \mid x \in H\},$$

and that the weak topology is defined by the family $\mathcal{P}_{\text{weak}}$ of seminorms given by

$$\mathcal{P}_{\text{weak}} = \{T \mapsto |\langle Tx, y\rangle| \mid x, y \in H\}.$$

(Recall that this topology does not coincide with the weak topology on $B(H)$ when viewing $B(H)$ as a Banach space.)

(i) Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $B(H)$ converging to some operator $T \in B(H)$ in the strong topology. Show that there exists a $C > 0$ such that $\sup_{n\in\mathbb{N}} \|T_n\| \leq C$.

(ii) Let $\mathcal{F}$ be a set in $B(H)$, and define its commutant $\mathcal{F}'$ by

$$\mathcal{F}' = \{S \in B(H) \mid ST = TS \text{ for all } T \in \mathcal{F}\}.$$

Show that $\mathcal{F}'$ is closed in $B(H)$ with respect to the weak topology on $B(H)$.

Exercise 2.

Let H be a Hilbert space, and let K be a closed linear subspace of H. Prove that any continuous linear functional $f\colon K \to \mathbb{C}$ has a **unique** continuous linear extension $\tilde{f}\colon H \to \mathbb{C}$ with the same norm, i.e., $\tilde{f}|_K = f$ and $\|\tilde{f}\| = \|f\|$. Do this in two steps.

(i) Let $P_K\colon H \to K$ denote the orthogonal projection of H onto K. Given an f as above, show that $\tilde{f}\colon H \to \mathbb{C}$ given by $\tilde{f}(x) = f(P_K(x))$ is a continuous linear extension such that $\|\tilde{f}\| = \|f\|$.

(ii) Show that $\tilde{f}$ is the unique continuous linear extension of f with the same norm.
Hint: Use the Riesz Representation Theorem.

Exercise 3.

Let X be a Banach space, and denote by $\iota\colon X \to X^{**}$ the isometry given by $\iota(x)\colon X^* \to \mathbb{C}$, $\omega \mapsto \omega(x)$ for all $x \in X$. Prove that the image $\iota((X)_1)$ of the unit ball $(X)_1$ of X is weak* dense in the unit ball $(X^{**})_1$ of X^{**}.

This is Exercise 1 of Chapter 8. You can find a hint there.

Exercise 4.

Recall that a countable group is amenable if and only if it admits a Følner sequence (see p. 96 of the lecture notes).

Recall also that a generating set S of a group G is a subset of the group such that every $g \in G$ can be written as $g = s_1 s_2 \dots s_n$ for some $s_1, s_2, \dots, s_n \in (S \cup S^{-1})$. A group is said to be finitely generated if we can choose a generating set consisting of finitely many elements.

(i) Consider the group $\mathbb{Z}$. Show that the sequence $(F_n)_{n\in\mathbb{N}}$ given by

$$F_n = \{0, 1, \dots, n-1\}$$

is a Følner sequence, and conclude that the group is amenable.

(ii) Using the Følner sequence above, explain how to define a Følner sequence for the group $\mathbb{Z}^k$ with $k \geq 2$.

(iii) Let G be a finitely generated group, and let S be a finite generating set of G. Suppose that for every $\varepsilon > 0$ there exists a finite subset $F \subset G$ such that

$$|sF \triangle F| < \varepsilon |F| \qquad \text{for all } s \in S.$$

(Note that the above inequality only has to hold for $s \in S$.)

Prove that G is amenable, by proving that G admits a Følner sequence.