Exam Probability Measure

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1. (a) Suppose that G is a countable group, with the σ-algebra 2^{Ω}. Describe in full detail all the translation-invariant measures, i.e., if $\mu : G \to [0, \infty]$ is a measure, then for every $A \subseteq G$, $\mu(A) = \mu(gA)$ for all $g \in G$.

   (b) Consider the measure space $\mathbb{Q}$ with its Borel-σ-algebra. Does there exist something like a "*Lebesgue measure on* $\mathbb{Q}$*?*". Justify your answer!

   If needed, explain what goes wrong in the construction analogous to the construction of the Lebesgue measure on the real line.

2. Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable subset with $\lambda(A) < \infty$. Show that for every $\varepsilon > 0$, there exist pairwise disjoint half-open intervals $I_1, \dots, I_n \subseteq \mathbb{R}$ such that for $E = \bigcup_{k=1}^{n} I_k$, it holds that $\lambda(A \Delta E) \leq \varepsilon$.

   (**Hint:** Remember the definition of the Lebesgue outer measure.)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function.

   (a) Show for every $\varepsilon > 0$ that there exists a compactly supported continuous $h : \mathbb{R} \to \mathbb{R}$ with

   $$\int_{\mathbb{R}} |f - h| d\lambda \leq \varepsilon$$

   (**Hint:** Assume first that f is a characteristic function on a Lebesgue measurable set A. You can use the result of this claim in the second part of the problem.)

   (b) Prove the following limit equality:

   $$\lim_{t \to 0} \int |f(x) - f(x+t)| d\lambda(x) = 0$$

4. Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability measure space and let $A_n \subseteq \mathcal{M}$ be a sequence of independent events. We define the following real random variable $Y_n = \frac{1}{n} \sum_{k=1}^{n} \chi_{A_k}$ as well as the probability average $p_n = \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(A_k)$. Show that $Y_n - p_n$ converges to zero in probability.

   Is this a special case of the weak law of large numbers? Explain! If it is relevant, describe which assumptions need to be added to make it a special case of the weak law of large numbers.