	Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q. 6	Q. 7	TOTAL (out of 20)
Points								

(Leave this table blank)

Your name:

Marco Zambon
Differential Geometry

Master in Mathematics, 2021-22

January exam

1. [4 points] For each real number $r > 0$, consider the map

$$\phi_r : \mathbb{R} \to \mathbb{R}$$

defined as follows: $\phi_r(t) = t$ if $t \leq 0$ and $\phi_r(t) = rt$ if $t > 0$.

a) For every r, show that the atlas $\{(\mathbb{R}, \phi_r)\}$ defines a structure of differentiable manifold on $\mathbb{R}$.

b) Given $r_1, r_2 > 0$, when are the corresponding manifolds diffeomorphic?

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2. [4 points] Let M be a manifold, f a smooth function on M, and X a vector field on M such that the function $X(f)$ does not vanish at any point of M.

a) Prove or disprove: if M is compact, then there exists no smooth function f and no vector field X on M with the above properties.

b) Prove or disprove: if γ is an integral curve of X, and $p \in M$ and $q \in M$ are distinct points lying on the the image of γ, then necessarily

$$f(p) \neq f(q).$$

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3. [4 points] Let $f\colon M \to N$ be a submersion.

a) Is it true that for all $c \in N$, the preimage $f^{-1}(c)$ is a submanifold of M? Explain.

b) Prove or disprove: the collection of connected components of the preimages $f^{-1}(c)$, as c ranges through all points of N, is a foliation on M.

Remark: Recall that f is a submersion if for every $p \in M$, the derivative $(f_*)_p$ is surjective.

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4. [2 points] Let M be a manifold, and $f \in C^\infty(M)$ be a function vanishing at some point $p \in M$. Prove or disprove: for all vector fields X, Y on M, the vector field

$$[f^2 X, Y]$$

vanishes at the point p.

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5. [1 point] Let M be a manifold of dimension ≥ 1. Consider the vector bundles TM (the tangent bundle) and $\mathbb{R}^2 \times M$ (the product vector bundle). Give an example (different from the zero map) of vector bundle map

$$\mathbb{R}^2 \times M \to TM.$$

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6. [3 points] For all natural numbers $n \geq 1$, compute the de Rham cohomology of $\mathbb{R}^n \setminus \{0\}$.

Further, for each integer k such that $H^k_{dR}(\mathbb{R}^n \setminus \{0\}) \neq \{0\}$, describe (as explicitly as you can) closed differential forms

$$\omega_1, \ldots, \omega_{i_k} \in \Omega^k(\mathbb{R}^n \setminus \{0\})$$

such that $[\omega_1], \ldots, [\omega_{i_k}]$ constitutes a basis of $H^k_{dR}(\mathbb{R}^n \setminus \{0\})$.
Remark: Here $\mathbb{R}^n \setminus \{0\}$ denotes $\mathbb{R}^n$ with the origin removed.

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7. [2 points] Consider the Lie algebra $\mathfrak{g} = (\mathbb{R}^2, [\ ,\ ] = 0)$. Exhibit three non-isomorphic Lie groups whose Lie algebra is $\mathfrak{g}$.