Commutative Algebra Exam - 2022/23

16th of January 2023

Question 1. What does it mean that a covariant functor $F : \mathcal{F} \to \mathcal{D}$ between two categories is left exact? Is the functor $\mathrm{Hom}_R(M, -)$ from Mod_R to itself left or right exact? **[2 pts.]**

Question 2. Explain what a local property of an R-module M is. List all te local properties of modules, rings, or morphisms seen during the course. **[2 pts.]**

Exercise 1. Let R be a ring and $I \subseteq R$ an ideal. Prove

a) If $J_1, J_2 \subseteq R$ are ideals of R such that $I \subseteq J_1 \cup J_2$, then $I \subseteq J_i$ for some $i = 1, 2$. **[1 pt.]**

b) If $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subseteq R$ are prime ideals of R such that $I \subseteq \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$, then $I \subseteq \mathfrak{p}_i$ for some $i = 1, \dots, n$. (*Hint: use induction on the number n of prime ideals.*) **[2 pts.]**

c) If R as a UFD with an infinite number of maximal ideals, then R has an infinite number of principal prime ideals. **[2 pts.]**

Exercise 2. Let $\{x_n\}_{n\in\mathbb{N}}, x_n \in \mathbb{C}$, be a sequence of complex numbers. Consider

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_0 x_n = 0, \quad \forall n \in \mathbb{N}, \quad a_i \in \mathbb{C}, \tag{1}$$

a homogeneous linear recurrence relation with constant coefficients. It can be shown that the set of solutions S of (1) is a finite dimensional complex vector space.

a) Let $T(\{x_n\}_{n\in\mathbb{N}}) = \{x_{n+1}\}_{n\in\mathbb{N}}$ be the translation operator. Show that S has the structure of a $\mathbb{C}[T]$-module and

$$S \cong \frac{\mathbb{C}[T]}{(T-\alpha_1)^{m_1}} \oplus \frac{\mathbb{C}[T]}{(T-\alpha_2)^{m_2}} \oplus \cdots \oplus \frac{\mathbb{C}[T]}{(T-\alpha_r)^{m_r}}.$$

where $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are the roots of the polynomial $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0$ with multiplicities $m_1, \dots, m_r \in \mathbb{N}$. **[2 pts.]**

b) Prove that the vector space of solutions S of any homogeneous linear recurrence relation (1) is generated by the sequences

$$\{\alpha_i^n\}_{n\in\mathbb{N}}, \{n\alpha_i^{n-1}\}_{n\in\mathbb{N}}, \dots, \{n^{m_i-1}\alpha_i^{n-m_i+1}\}_{n\in\mathbb{N}}, \quad \text{for all } i = 1, \dots, r.$$

As an application, give a general formula for the n-th Fiboncci number F_n. Recall that $F_{n+2} = F_{n+1} + F_n$ with $F_0 = F_1 = 1$. **[2 pts.]**

Exercise 3. (Related to the assignments) Let R be a ring and M an R-module. Let $\operatorname{Ass} M$ be the set of associated primes of M.

a) Prove that $\mathfrak{p} \in \operatorname{Ass} M$ if and only if there is an injection $\iota : R/\mathfrak{p} \hookrightarrow M$ of R-modules. **[1 pt.]**

b) If $N \subset M$ is an R-submodule, show that $\operatorname{Ass} M \subset \operatorname{Ass} N \cup \operatorname{Ass} M/N$. **[1 pt.]**

c) Show that for a prime ideal $\mathfrak{p}$ of R, $\operatorname{Ass} R/\mathfrak{p} = \{\mathfrak{p}\}$. Deduce that, if R is Noetherian and M is finitely generated, the set $\operatorname{Ass} M$ of associated primes of M is finite. **[1 pt.]**

d) Let $\mathrm{ZD}(M) = \{r \in R \mid \exists m \in M \setminus \{0\}, rm = 0\}$ denote the set of zero divisors of M. If R is Noetherian, show that

$$\mathrm{ZD}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

[2 pts.]

e) Let $(R, \mathfrak{m})$ be a Noetherian local ring. If $\mathfrak{m} \setminus \mathfrak{m}^2$ consists entirely of zero divisors, prove there is a nonzero $x \in R$ such that $x\mathfrak{m} = 0$. (*Hint: consider elements of the form* $a + y^n$ *with* $a \in \mathfrak{m} \setminus \mathfrak{m}^2, y \in \mathfrak{m}, n \in \mathbb{N}$) **[2 pts.]**