Commutative Algebra Exam - January 2025

Question 1

Name all the universal properties discussed in the course. State the universal property of the quotient of a set by an equivalence relation, no proof is required. **[2 pts]**

Question 2

Let R be a ring. Find a condition on R for which submodules of free R-modules are also free. Give a counterexample for rings not satisfying this condition. **[2 pts]**

Problem 1

Let M be a flat R-module. Let $r \in R$ be a non-zero divisor. Show that for any $m \in M$, if $rm = 0$, then $m = 0$. **[2 pts]**

Problem 2

Let M be an R-module and let $I \subseteq R$ be an ideal. The I-torsion $\Gamma_I(M)$ of M is defined as the set

$$\Gamma_I(M) = \{m \in M \mid \exists n \in \mathbb{N}, I^n m = 0\}.$$

a) Show that the I-torsion defines a functor $\Gamma_I \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$. **[2 pts]**

b) Prove that Γ_I is a left exact functor. **[2 pts]**

c) Assume that R is Noetherian and let $S \subseteq R$ be a multiplicative set. Show that Γ_I commutes with the localisation functor $S^{-1} \colon \mathrm{Mod}_R \to \mathrm{Mod}_{S^{-1}R}$. **[2 pts]**

Problem 3

Let (R, m) be a Noetherian local ring. Define the ideal

$$I = \cap_{k \geq 0} m^k.$$

a) Prove there is an ideal $J \subseteq R$ with the property $J \cap I = mI$ that is maximal with respect to inclusion and such that for any $f \in m$, there exists $\alpha \in \mathbb{N}$ satisfying $(J \colon f^\alpha) = (J \colon f^{\alpha+1})$. **[2 pts]**

b) For any $f \in m$, show that $f^\alpha \in J$. **[3 pts]**

c) Prove that $m^n \subseteq J$ for some $n \in \mathbb{N}$, and deduce that $I = 0$. **[3 pts]**