Exam Differential Geometry - January 2025

1. **[3 points]**
   Let n be a positive integer and $k \in \{0, \dots, n\}$. Consider the map

$$F_k \colon \mathbb{R}^n \to \mathbb{R}, (x_1, \dots, x_n) \mapsto \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2.$$

   a) For which values of k is $F_k^{-1}(1)$ a submanifold of $\mathbb{R}^n$?

   b) For which values of k is $F_k^{-1}(1)$ a compact submanifold of $\mathbb{R}^n$?

2. **[3 points]**
   On $\mathbb{R}^2$ consider the vector field $X = (x + y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Fix a point $(x_0, y_0) \in \mathbb{R}^2$. Denote by γ the integral curve of X satisfying $\gamma(0) = (x_0, y_0)$.

   a) Compute explicitly the integral curve γ.

   b) Show that $\lim_{t \to -\infty} \gamma(t) = 0$ (This question was wrong and therefore removed)

3. **[4 points]** Let M be a manifold, N a submanifold of codimension 1. Assume that there is a section Y of the vector bundle $TM|_N \to N$ which is no-where tangent to N, i.e. $Y_p \notin T_pN$ for all $p \in N$.

   a) If M is orientable, does it follow that N is an orientable manifold?

   b) If N is an orientable manifold, does is follow that M is orientable too?

   Remark: $TM|_N \to N$ denotes the restriction to N of the tangent bundle of M.

4. **[6 points]**

   a) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a Lie ideal of $\mathfrak{g}$, i.e. $[x, v] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and $v \in \mathfrak{h}$. Does there exist a Lie algebra structure on $\mathfrak{g}/\mathfrak{h}$ such that the projection

$$\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}, x \mapsto x \text{ mod } \mathfrak{h}$$

   is a Lie algebra map?
   Now let G be a connected Lie group, let H be a connected normal Lie subgroup.

   b) Provide an example showing that H is not necessarily a closed subset of G.

   c) Prove or disprove: when G is simply connected, H has to be a closed subset of G.

   Remark: Recall that a subgroup H is normal iff $ghg^{-1} \in H$ for all $g \in G$, $h \in H$.
   Hint: Denote by $\mathfrak{g}$ the Lie algebra of G an be $\mathfrak{h}$ the Lie algebra of the normal Lie group H. You can use (without needing to prove it) that $\mathfrak{h}$ is a Lie ideal of $\mathfrak{g}$.

5. **[2 points]**
   On $\mathbb{R}^3$, consider that 1-form $\alpha = f dx$ where $f \in C^\infty(\mathbb{R}^3)$.

   a) Which are the functions f for which α is a closed 1-form?

   b) Which are the functions f for which α is an exact 1-form?

6. **[2 points]**
   Given a manifold M, state the relation between foliations of M and involutive distributions on M. you do not need to give a proof.