# Exam Functional Analysis - January 2025

## Problem 1

We denote by $H$ the Hilbert space $L^2([1,2],\lambda)$ where $\lambda$ is the Lebesgue measure on $[1,2]$. Define $T\colon H \to H$ by $(Tf)(x) = \frac{f(x)}{x}$

(a) Show that $T$ is a bounded operator.

(b) Show that $T$ does not have eigenvalues.

(c) Prove that $\sigma(T) = [1/2, 1]$. **Hint:** let $t \in ]1,2[$ and define $g(x) = (xt)^{-1}$ for $x \in [1,2]$. Prove that if $(T - t^{-1}I)f = g$ for some function $f\colon [1,2] \to \mathbb{C}$ then $f \notin L^2([1,2],\lambda)$.

(d) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $H$ and let $S \in B(H)$ satisfy $Se_n = \frac{1}{n+1}e_{n^2}$. Prove that $H$ admits an orthonormal basis of eigenvectors of $S^*TS$.

## Problem 2

Let $H$ be a Hilbert space and let $T \in B(H)$ be a self-adjoint operator satisfying $T^3 = T$. Prove that $\sigma(T) \subseteq \{-1, 0, 1\}$.

## Problem 3

Let $X$ be an infinite dimensional Banach space.

(a) Argue that the unit ball $X_1 = \{x \in X \mid ||x|| \le 1\}$ is closed in the weak topology.

(b) Prove that the closure of the set $\{x \in X \mid ||x|| = 1\}$ in the weak topology is $X_1$.
**Hint:** Since $X$ is infinite dimensional the kernel of any linear map $X \to \mathbb{C}^n$ must be non-trivial.

(c) Let $B \subset X^* \setminus (X^*)_1$ be a non-empty convex set. Assume that for every $\psi \in X^{**}$ and every $\varepsilon > 0$ there exists $\omega \in X^*$ and $b \in B$ with $||\omega|| \le 1$ and $|\psi(\omega - b)| < \varepsilon$. Prove that $B$ is not closed in the weak* topology.

## Problem 4

Let $X$ and $Y$ be Banach spaces.

(a) Let $T\colon X \to Y$ be a surjective linear map bounded from below, i.e. there exists a $\delta > 0$ such that $||Tx|| \ge \delta||x||$. Prove that $T$ is continuous.

(b) Assume that $f\colon X \times Y \to \mathbb{C}$ is a map such that for each $x \in X$ the map $y \mapsto f(x,y)$ is bounded and linear and for each $y \in Y$ the map $x \mapsto f(x,y)$ is bounded and linear. Prove that there exists a $k > 0$ such that $|f(x,y)| \le k||x|| \cdot ||y||$ for all $x \in X$ and $y \in Y$.