Early Universe Cosmology: Exam 15 June 2021

June 16, 2021

1 Questions by Hertog

We were given some formulae on a separate sheet of paper, like the Boltzmann equations for perturbations and the two relevant Einstein equations, the number density in equilibrium (in integral form), the Boltzmann equation from Chapter 3 from Baumann, P_Φ from inflation ...

1.1 Recombination (8pts)

Around 380,000 years after the Big Bang, protons and electron combined to form hydrogen atoms, an epoch called *recombination*.

(i) Given the interaction rate Γ for the process $e^- + p^+ \leftrightarrow H + \gamma$, describe how we can determine when the reaction is in thermal equilibrium in the context of an expanding universe.

(ii) Show that, using the Boltzmann equation in equilibrium (the Saha equation), the electron fraction X_e prior to recombination is given by

$$\frac{X_e^2}{1 - X_e} = \frac{\pi^2}{2\zeta(3)}\eta \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-\varepsilon_0/T} \,, \tag{1.1}$$

where $\varepsilon_0 = m_e + m_p - m_H$ is the binding energy of H. You can assume that the universe is neutral and that $n_b = n_p + n_H$. (Masses and the value of ϵ_0 were given).

(iii) Define the temperature of recombination T_{rec} such that $X_e(T_{rec}) = 10^{-1}$. Show that $T_{rec} \sim 0.3$ eV. You can use that $\ln(10^{17}) \sim 40$. Why is this much lower than the value of ϵ_0?

(iv) Discuss qualitatively electron freeze-out and photon decoupling, two important consequences of recombination. Make a sketch of $X_e(T)$ (from high to low temperature) and clearly denote T_{rec} and T_{dec} (temperature of photon decoupling) in your plot. Also show the prediction by the Saha equation.

1.2 Big Bang and Quantum Cosmology (4pts)

Given that the matter content of the universe always satisfied the strong energy condition $\rho + 3P \geq 0$, prove that the universe always had a Big Bang in the past at a time t ($a(t) = 0$) such that $t_0 - t < 1/H_0$. How does quantum cosmology solve this problem? Describe what form the solution would take, and what we might learn from this (max 1/2 page).

1.3 Homework: Meszaros (3+2pts)

(i) Derive the Meszaros equation (was given) from the Boltzmann equations and the Einstein equations (neglecting Θ_0, Θ_1, and assuming $k/(aH) \gg 1$).

(ii) Verify that

$$\delta_1 \propto y + \frac{2}{3} \tag{1.2}$$

$$\delta_2 \propto \left(y + \frac{2}{3}\right) \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 2\sqrt{1+y} \tag{1.3}$$

are solutions to the Meszaros equation. Also discuss their late-time behaviour.

(iii) Define the transfer function $T(k)$ and discuss how it behaves for large and small k. Sketch the matter power spectrum for a Harrison-Zel'dovich-Peebles spectrum, and discuss what changes if we add a cosmological constant.

2 Questions by Craps

We were given some formulae like the equation of motion of the inflaton field, the definition of the slow-roll parameters, the first Friedmann equation,...

2.1 Homework: slow-roll parameters (2pts)

Show that, to first order in the perturbation variables, we have

$$\epsilon = \frac{1}{8\pi G}\left(\frac{V'}{V}\right)^2 . \tag{2.1}$$

2.2 Regular exam questions

You were expected to write a few sentences for each question, and each question counted for 1.5 pts.

(i) Why do the simplest inflation models predict nearly Gaussian-like perturbations?

(ii) Given a field $f(\mathbf{x})$ and its Fourier expansion with modes $f_{\mathbf{k}}$, with $f_{\mathbf{x}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$, and given the distribution $p(a_{\mathbf{k}}, b_{\mathbf{k}})$, argue why p is homogeneous and isotropic.

(iii) We saw that for the temperature anisotropy ina given direction, the integrated Sachs-Wolfe term is given by the integral

$$\frac{\delta T}{T}(\hat{p}) = \int_{\eta_{dec}}^{\eta_0} \left(\dot{\psi} - \dot{\phi} \right) d\eta \,. \tag{2.2}$$

Explain all the symbols in the right hand side. Naively, one might think that the integral over a derivative reduces to the contribution from the two boundaries. Discuss why this is not the case.

(iv) Discuss the significance of the polarization spectra to determine the optical depth to recombination.