Functional Analysis Fall 2020, Final exam

29 January 2021, 13:00–16:00

Each problem is worth 4 points. The best 3 out of 4 solutions will count towards the final grade.

Problem 1. Define a function h_n on $[0,1)$ in the following way: if $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for some $k \in \{0, 1, \dots, 2^n-1\}$ then $h_n(x) = 1$ if k is even and $h_n(x) = -1$ if k is odd. Recall that $L^\infty[0,1) \simeq (L^1[0,1))^*$. Check if the sequence $(h_n)_{n\in\mathbb{N}}$ converges in the weak* topology of $L^\infty[0,1)$. If it does, compute its limit.

Problem 2. Let $(h_n)_{n\in\mathbb{N}}$ be the sequence from the previous problem. Does it converge in the weak topology of $L^\infty[0,1)$? If it does, identify its limit.

Problem 3. Let $H = L^2(\mathbb{R})$. Define on H an operator $(Tf)(x) := f(x+1) + f(x-1)$. Prove that T is a bounded, self-adjoint operator. Compute its spectrum.

Problem 4. Let X and Y be Banach spaces. Suppose that a linear map $T : X \to Y$ is continuous when X and Y are equipped with their weak topologies. Show that T is a bounded operator.

Good luck!