Exam Introudction to General relativity

30 january 2018

1. (a) Explain briefly but accurately the difference between free motion in Newtonian mechanics and general relativity.
   (b) The line element of the wormhole geometry is given by
   $$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2).$$
   Write down the geodesic equation. Compute how much proper time a radially moving traveller starting with proper radial velocity $u_r = U$ at $r = R$ takes to get from $r = R$ to $r = -R$.

2. The generalisation of the Schwarzschild geometry for a cosmological constant Λ is given by ($G = c = 1$)
   $$ds^2 = -(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})dt^2 + (1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})^{-1}dr^2 + r^2 d\Omega_2^2$$
   with $d\Omega_2^2$ the line element of the 2-sphere.
   (a) Derive an equation for $r(\lambda)$ of timelike geodesics with λ an affine parameter in terms of an effective potential $V_{eff}(r)$.
   (b) How does a nonzero (positive or negative) constant modify the bound orbits of massive particles?
   (c) Set $\Lambda = 0$ ans sketch the qualitative behavior of a particle coming in at infinity with energy ϵ equal to the maximum of the effective potential. How much does this change if ϵ is a bit larger or a bit smaller? Do you know a relevant physical situation?
   (d) Again set $\Lambda = 0$. What is the longest proper time one can spend across the event horizon before being destroyed in the singularity?
   (e) With $\Lambda = 0$, consider an observer falling radially inward with zero kinetic energy at infinity. How much time does it take to pass between $6M$ and $2M$.

3. The line element for the RW universe is
   $$ds^2 = -dt^2 + a(t)^2(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2)$$
   with $k = 1, 0, -1$ for respectively closed, flat or open universes. The Friedman-Lemaitre equation describes the evolution of $a(t)$ in an RW universe
   $$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - k$$
   with ρ the density.
   (a) Rewrite this in terms of an effective potential $U_{eff}(a)$ for the scale factor and show that there is a critical value of ρ_m for which a does not evolve in time. Find this value. What is the spatial volume in terms of Λ?
   (b) Illustrate with a causal diagram the notion of a horizon in cosmology. Derive an expression for the physical distance $d_{hor}(t)$ in a flat matter-dominated universe. Compute the age of the universe in terms of H_0 using the current value of $72Mpc$.This gives $t_0 = 9Gyr$. How come this is less than the age of some galaxies?
   (c) Show that in FLRW models that if $\rho + 3p$ is always positive, then there will always be a singulrity at some time in the past. Is this the case in our universe?