1 Black holes (8pts)

The Schwarzchild geometry is given by the following metric (in units with $G = c = 1$):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1}$$

The transformation from Schwarzchild coordinates (t, r, θ, ϕ) to Kruskal coordinates (U, V, θ, ϕ) defined for $r > 2M$ by

$$U = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{r/4M} \cosh\left(t/4M\right), V = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{r/4M} \sinh\left(t/4M\right) \tag{2}$$

with $\left(\frac{r}{2M} - 1\right) e^{r/2M} = U^2 - V^2$

a) Draw a Kruskal diagram and indicate the singularity and the horizon as well as the world lines of an in falling and a distant observer. Show that in a Kruskal diagram $\left|\frac{dV}{dU}\right|$ must be greater than unity for a timelike particle world line even if it is moving non-radially.

b)Explicitly carry out the transformation to Penrose coordinates (u', v') and construct the Penrose diagram, a compact version of the Kruskal diagram, indicating the coordinate ranges for the Penrose coordinates (u', v'), the different kinds of infinities, the location of the event horizon and singularity as well as the worldlines of an infalling and distant observer.

c) Show that the normal vector to the horizon three-surface of a Schwarzchild black hole is a null vector.

2 Gravitational lensing (4pts)

Consider a gravitational lens with $D_L = D_{LS} = D_S/2$ and $\beta << \theta_E << 1$

a)Draw a figure depicting this lensing situation, the angles, the two paths that light can take for the source S to observer O.

b) Derive the difference in path length, to first order in β, in terms of the Einstein angle and the basic parameters of the lens configuration.

3 Cosmology (5pts)

a) In the course the redshift of a photon in a Schwarzchild geometry has been calculated using time-translation symmetry. Derive in a similar way the cosmological redshift from space translation symmetry of the Flat RW geometry with line-element:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{3}$$

b)Derrive an approximate solution for a(t) from the Friedmann-Lemaître equation in a universe (3) that contains a mix of radiation, matter and vaccuum energy in ratios similar to what we observe. Sketch your result and indicate a number of important event in the history of our universe.

4 Gravitational waves (3pts)

$$ds^2 = -dt^2 + (1 + f(t - z))dx^2 + (1 - f(t - z))dy^2 + dz^2 \quad (4)$$

describes a plane gravitational wave propagating in the z direction.
a)Explain why it is impossible to detect a gravitational wave with a single test mass.
b)Consider two test masses one in the origin and one at location (x,y,z) in the Cartesian coordinates used in (4). Derive the change in distance between the masses produced by the gravitational wave.