Exam Probability & Measure **(version A)**

January 19, 2018 (13u00)

NAME: ....

---

1. We consider $\mathbb{R}$ with its standard measure structure (i.e., Borel σ-algebra and Lebesgue measure λ). Let $f \in \mathcal{L}^1(\mathbb{R})$ and consider the function

$$\varphi : \mathbb{R} \to \mathbb{C} : x \mapsto \varphi(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + x^2 + y^2}\, d\lambda(y).$$

   (a) Is φ continuous? Explain.

   (b) Is φ Lebesgue integrable over $\mathbb{R}$? Explain.

2. Let μ_1 and μ_2 be two finite positive measures on a measurable space $(\Omega, \mathfrak{M})$. Consider the real measure $\mu = \mu_1 - \mu_2$.

   (a) Can one conclude that $|\mu| = \mu_1 + \mu_2$?

   (b) Suppose there exists some $A \in \mathfrak{M}$ such that $\mu_1(A^c) = \mu_2(A) = 0$. Compute the measures $\mu^{\pm}$ appearing in the Jordan decomposition of μ.

   (c) After making (b), what conjecture is now tempting to make? Formulate it precisely and examine whether the conjecture really holds.

3. Consider the probability space $(\Omega, \mathfrak{M}, \mathbf{P})$ where $\Omega = [0,1] \times [0,1]$, $\mathfrak{M}$ is the standard Borel σ-algebra on Ω, and $\mathbf{P}$ is the Lebesgue measure. Consider the random variables $X : \Omega \to \mathbb{R} : (x,y) \mapsto x$ and $Y : \Omega \to \mathbb{R} : (x,y) \mapsto y$. Denote with $X \wedge Y$ the function from Ω to $\mathbb{R}$ given by $(X \wedge Y)(x,y) = \min\{x,y\}$.

   Compute $\mathbf{E}(X \mid X \wedge Y)$. Verify using the definition of conditional expectation that your result is correct.

4. Suppose that $(\varphi_n)_n$ is a sequence of characteristic functions (of real random variables) that converges pointwise to a function $\psi : \mathbb{R} \to \mathbb{C}$ which is continuous in 0. Can one conclude that ψ is a characteristic function (of some real random variable)? Explain.

Success !

Exam Probability & measure **(version B)**

January 19, 2018 (13u00)

NAME: ....

---

1. We consider $\mathbb{R}$ with its standard measure structure (i.e., Borel σ-algebra and Lebesgue measure λ). Let $f \in \mathcal{L}^1(\mathbb{R})$ and consider the function

$$\varphi : \mathbb{R} \to \mathbb{C} : x \mapsto \varphi(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + x^2 + y^2} \, d\lambda(y).$$

   (a) Is φ continuous? Explain.

   (b) Is φ Lebesgue integrable over $\mathbb{R}$? Explain.

2. Let μ_1 and μ_2 be two finite positive measures on a measurable space $(\Omega, \mathfrak{M})$. Consider the real measure $\mu = \mu_1 - \mu_2$.

   (a) Can one conclude that $|\mu| = \mu_1 + \mu_2$?

   (b) Suppose there exists some $A \in \mathfrak{M}$ such that $\mu_1(A^c) = \mu_2(A) = 0$. Compute the measures $\mu^{\pm}$ appearing in the Jordan decomposition of μ.

   (c) After making (b), what conjecture is now tempting to make? Formulate it precisely and examine whether the conjecture really holds.

3. Consider the probability space $(\Omega, \mathfrak{M}, \mathbf{P})$ where $\Omega = [0,1] \times [0,1]$, $\mathfrak{M}$ is the standard Borel σ-algebra on Ω, and $\mathbf{P}$ is the Lebesgue measure. Consider the random variables $X : \Omega \to \mathbb{R} : (x, y) \mapsto x$ and $Y : \Omega \to \mathbb{R} : (x, y) \mapsto y$. Denote with $X \wedge Y$ the function from Ω to $\mathbb{R}$ given by $(X \wedge Y)(x, y) = \min\{x, y\}$.

   Compute $\mathbf{E}(X \mid X \wedge Y)$. Verify using the definition of conditional expectation that your result is correct.

4. Let G be a locally compact group. Let λ be 'the' left and ρ 'the' right Haar measure on G. Can one conclude, in general, that λ and ρ are absolutely continuous w.r.t. each other? If not, illustrate this by providing an explicit example; if so, prove that result and determine the Radon-Nikodym derivatives.

Success !