# Exam Probability & Measure (version A)

*September 3, 2018 (13u00)*

NAME: ..........................................................

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1. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in $\mathcal{L}^1(\Omega, \mathfrak{M}, \mu)$. Suppose there exists some $p \in \mathbb{R}_0^+$ such that
$$\int_\Omega |f_n|\, d\mu \leq \frac{1}{n^p} \quad \text{for all } n \in \mathbb{N}_0.$$
Can one conclude that $f_n(x) \to 0$ for $\mu$-almost all $x \in \Omega$?

2. Let $(\Omega, \mathfrak{M}, \mathbf{P})$ be the probability space where $\Omega = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, y \geq 0,\ x^2 + y^2 \leq 1\big\}$, $\mathfrak{M}$ is the Borel-$\sigma$-algebra on $\Omega$, and $\mathbf{P}$ is the normalized Lebesgue measure. Consider the random variables
$$X : \Omega \to \mathbb{R} : (x, y) \mapsto x \quad \text{and} \quad Y : \Omega \to \mathbb{R} : (x, y) \mapsto y.$$
   (a) Is $\{X, Y\}$ an independent pair? Explain why or why not.
   (b) Compute $\mathbf{E}(Y \mid X^2 + Y^2)$. Verify, using the definition of conditional expectation, that your answer is correct.

3. Consider $\mathbb{R}$ with its standard measure structure (i.e., with the Borel-$\sigma$-algebra and the Lebesgue measure $\lambda$). Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue-integrable function. What would you conjecture about
$$\lim_{t\to 0} \int_\mathbb{R} \big|f(x - t) - f(x)\big|\, d\lambda(x)\ ? \qquad (*)$$
   (a) What result/technique seems, at first sight, appropriate to prove you conjecture? What difficulties do you face when trying to elaborate that idea into a rigourous proof?
   (b) Devise a stepwise strategy to actually prove your conjecture. Or do you begin to doubt the validity of your intial conjecture and are you thinking about the construction of some counterexample ...?

   *Spend at most 20 minutes to questions (a) and (b) and then give me a high sign. Then we will look together into the stage reached in your search process. If needed, I will give you some suggestions and directions for the continuation of your search process[1]. Of course, if you wouldn't need this interactive intermezzo, you can jump right away to (c).*

   (c) Now, what can you conclude about the limit in $(*)$? Argue rigourously!

4. Suppose that $(X_n)_n$ is a sequence of real-valued random variables all having finite mean and finite variance. Denote by $\mu_n$ the mean and by $\sigma_n^2$ the variance of $X_n$. Suppose the sequences $(\mu_n)_n$ and $(\sigma_n^2)_n$ are bounded.
Can one conclude there exists a real-valued random variable $X$ and a subsequence $(X_{n_k})_k$ converging in distribution to $X$?

*Success !*

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[1] For the evaluation of question 3, not only the final result but also the quality of your conjecturing and search process will be taken into account. So, if you wouldn't have reached the final result without using some hints, this does NOT imply that question 3 would be completely "lost".

# Exam Probability & measure (version B)

*September 3, 2018 (13u00)*

NAME: ..............................................................

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1. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in $\mathcal{L}^1(\Omega, \mathfrak{M}, \mu)$. Suppose there exists some $p \in \mathbb{R}_0^+$ such that
$$\int_\Omega |f_n|\, d\mu \leq \frac{1}{n^p} \quad \text{for all } n \in \mathbb{N}_0.$$
Can one conclude that $f_n(x) \to 0$ for $\mu$-almost all $x \in \Omega$?

2. Let $(\Omega, \mathfrak{M}, \mathbf{P})$ be the probability space where $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\ x^2 + y^2 \leq 1\}$, $\mathfrak{M}$ is the Borel-$\sigma$-algebra on $\Omega$, and $\mathbf{P}$ is the normalized Lebesgue measure. Consider the random variables
$$X : \Omega \to \mathbb{R} : (x, y) \mapsto x \quad \text{and} \quad Y : \Omega \to \mathbb{R} : (x, y) \mapsto y.$$
   (a) Is $\{X, Y\}$ an independent pair? Explain why or why not.
   (b) Compute $\mathbf{E}(Y \mid X^2 + Y^2)$. Verify, using the definition of conditional expectation, that your answer is correct.

3. Consider $\mathbb{R}$ with its standard measure structure (i.e., with the Borel-$\sigma$-algebra and the Lebesgue measure $\lambda$). Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue-integrable function. What would you conjecture about
$$\lim_{t\to 0} \int_{\mathbb{R}} \big|f(x - t) - f(x)\big|\, d\lambda(x)\ ? \qquad (*)$$
   (a) What result/technique seems, at first sight, appropriate to prove you conjecture? What difficulties do you face when trying to elaborate that idea into a rigourous proof?
   (b) Devise a stepwise strategy to actually prove your conjecture. Or do you begin to doubt the validity of your intial conjecture and are you thinking about the construction of some counterexample ...?

   *Spend at most 20 minutes to questions (a) and (b) and then give me a high sign. Then we will look together into the stage reached in your search process. If needed, I will give you some suggestions and directions for the continuation of your search process[1]. Of course, if you wouldn't need this interactive intermezzo, you can jump right away to (c).*

   (c) Now, what can you conclude about the limit in $(*)$? Argue rigourously!

4. Consider the set $G$ of all invertible upper triangular $(2 \times 2)$-matrices over $\mathbb{R}$, i.e.,
$$G = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \,\middle|\, (x, y, z) \in \mathbb{R}_0 \times \mathbb{R} \times \mathbb{R}_0 \right\}.$$
Then, $G$ is a locally compact group is for the multiplication of matrices and for the standard topology on $\mathbb{R}_0 \times \mathbb{R} \times \mathbb{R}_0$. Find the left Haar measure on $G$.

*Success !*

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[1] For the evaluation of question 3, not only the final result but also the quality of your conjecturing and search process will be taken into account. So, if you wouldn't have reached the final result without using some hints, this does NOT imply that question 3 would be completely "lost".