MA Physics January 2012

General Relativity

T. Hertog

1. A general static, spherically symmetric metric in d space-time dimensions can be written as

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_{d-2}^2, \tag{1}$$

where $d\Omega_{d-2}^2$ is the standard round metric on S^{d-2}, defined inductively by $d\Omega_1^2 = d\phi^2$ and $d\Omega_{i+1}^2 = d\theta_i^2 + \sin^2\theta_i d\Omega_i^2$ for $i \geq 1$, with $0 < \phi \leq 2\pi$ and $0 \leq \theta_i \leq \pi$. Assume that $A(r)$ and $B(r)$ are analytic functions of r such that both have a simple zero at $r = r_+ > 0$ and are positive for $r > r_+$.

(a) Show that radial null geodesics are given by $t \pm r^* =$constant, where

$$r^* = \int_{r_0}^{r} \frac{dx}{\sqrt{A(x)B(x)}}, \tag{2}$$

with $r_0 > r_+$ an arbitrary constant. Show that $r^* \to -\infty$ as $r \to r_+$.

(b) Obtain the metric in ingoing Eddington-Finkelstein coordinates. Explain why this metric can be analytically continued through $r = r_+$.

(c) The timelike Killing field is $k \equiv \partial/\partial t$ in static coordinates. Show that $k = \partial/\partial v$ in EF coordinates, and that $r = r_+$ is a Killing horizon of k. What is the surface gravity?

(d) Check that your formula gives the correct answer for the Schwarzschild solution.

(e) What happens if A and B both have a zero of order $p > 1$ at $r = r_+$ instead of a simple zero?

2. The generalization of the Schwarzschild solution to spacetimes with a cosmological constant Λ is given by the following metric (in units where $c = G = 1$),

$$ds^2 = -\left[1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right] dt^2 + \left[1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right]^{-1} dr^2 + r^2 d\Omega_2^2. \tag{3}$$

(a) Discuss briefly the causal structure of the Schwarzschild-de Sitter spacetime given by (3) with $\Lambda > 0$, e.g. by drawing the Kruskal diagram.

(b) Discuss the behavior of geodesics of massive particles in these spacetimes, in particular how a nonzero cosmological constant (either positive or negative) modifies the bound orbits of the Schwarzschild geometry.

3(a) Consider two well-separated, approximately Schwarzschild black holes at rest. Let their masses be M_1 and M_2. Assume that they coalesce to form a Schwarzschild black hole of mass M_3, with gravitational waves carrying off energy E in the process. Use conservation of energy and the second law of black hole mechanics to prove that the efficiency of this process, defined as $\eta = E/(M_1 + M_2)$ obeys $\eta \leq 1 - 1/\sqrt{2}$.
(b) Let E denote the maximum energy that can be extracted from a Kerr black hole in the Penrose process. The efficiency of this process is $\eta \equiv E/M$ where M is the initial mass of the black hole. Calculate what is the largest possible value of η.

4. Let E and L be the energy and angular momentum per unit mass of a neutral particle in free fall within the equatorial plane, i.e. on a timelike ($\sigma = 1$) or null ($\sigma = 0$) geodesic with $\theta = \pi/2$, of a Kerr black hole.
(a) Show that the particle's Boyer-Lindquist radial coordinate r satisfies (in units where $c = G = 1$)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = 0, \tag{4}$$

where λ is an affine parameter, and the effective potential V is given by

$$V(r) = -\sigma\frac{M}{r} + \frac{L^2}{2r^2} + \frac{1}{2}(\sigma - E^2)\left(1 + \frac{a^2}{r^2}\right) - \frac{M}{r^3}(L - aE)^2 \tag{5}$$

(b) For an extremal Kerr black hole with $a = M$, the last stable circular orbit of a corotating massive particle has $R_c = M$ and $E_c = 1/\sqrt{3}$. A particle initially in a circular orbit with much larger radius $r >> M$ (and thus $E \approx 1$) will emit gravitational radiation and should therefore slowly spiral in to smaller radii as it loses energy, remaining in an approximately circular orbit until it reaches the orbital radius R_c. The binding energy E_b per unit rest mass in the last stable circular orbit is given by $E_b = 1 - E_c$ and represent the fraction of the original rest energy that is radiated away during the time in which the particle spirals to R_c. Derive the analogous binding energy of the closest stable geodesic circular orbit in the Schwarzschild geometry and compare with extremal Kerr.