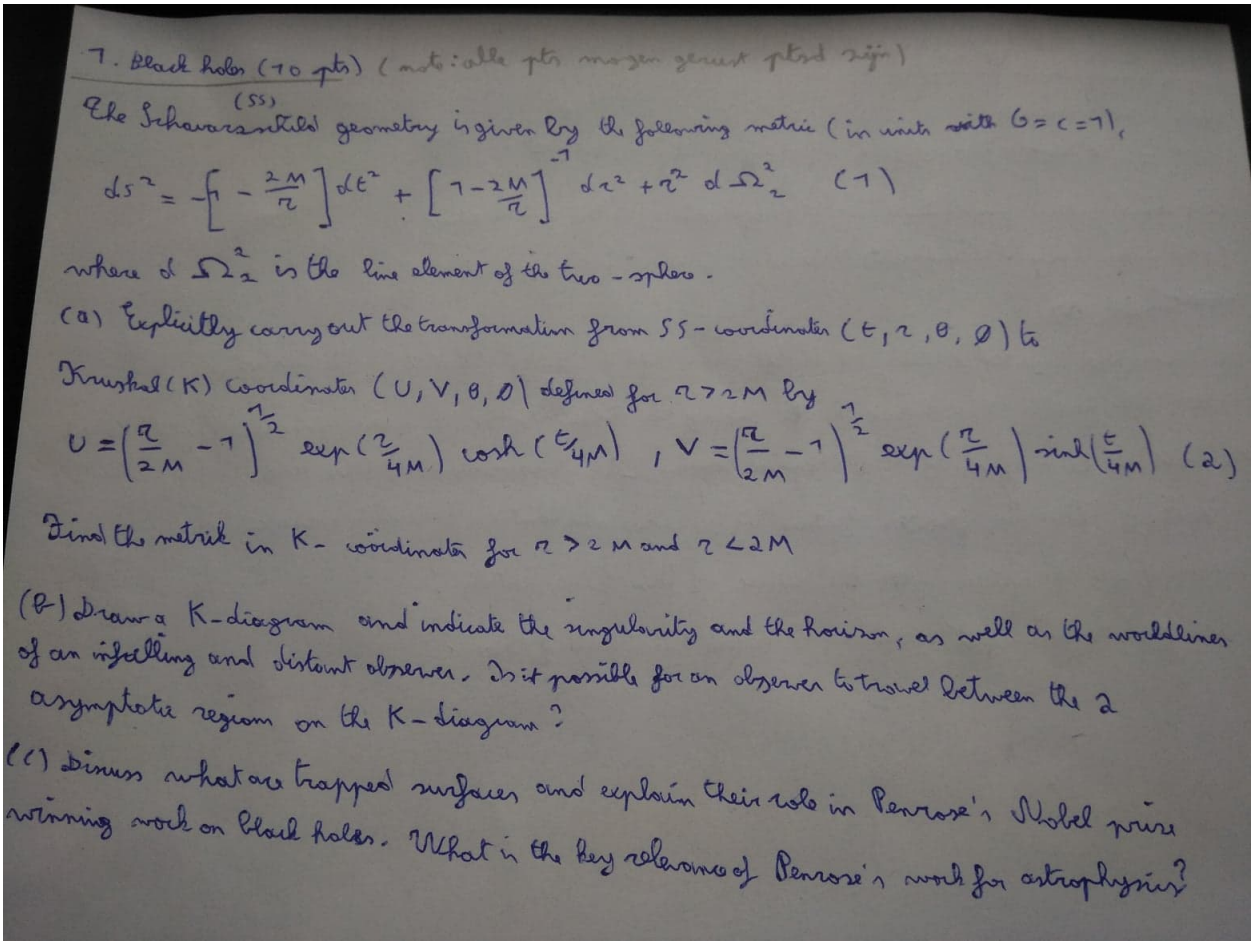
Exam introduction to GR 08/01/2021

If the pictures or my handwriting aren't clear, well suck it up buttercup, I really don't feel like typing all of it out right now and still want to help the people who didn't do the exam yet.



7. Black holes (10 pts) (note: alle pts moeten gemist gaat zijn)

The (SS) Schwarzschild geometry is given by the following metric (in units with $G = c = 1$),

$$ds^2 = -\left[1 - \frac{2M}{r}\right] dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2 d\Omega_2^2 \quad (1)$$

where $d\Omega_2^2$ is the line element of the two-sphere.

(a) Explicitly carry out the transformation from SS-coordinates (t, r, θ, ϕ) to Kruskal (K) coordinates (U, V, θ, ϕ) defined for $r > 2M$ by

$$U = \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \cosh\left(\frac{t}{4M}\right), \quad V = \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \sinh\left(\frac{t}{4M}\right) \quad (2)$$

Find the metric in K-coordinates for $r > 2M$ and $r < 2M$

(β) Draw a K-diagram and indicate the singularity and the horizon, as well as the worldlines of an infalling and distant observer. Is it possible for an observer to travel between the 2 asymptotic regions on the K-diagram?

(γ) Discuss what are trapped surfaces and explain their role in Penrose's Nobel prize winning work on black holes. What is the key relevance of Penrose's work for astrophysics?

(c) Discuss what are trapped surfaces and explain their role in Penrose's Nobel prize winning work on black holes. What is the key relevance of Penrose's work for astrophysics?

2. cosmology (5 pts)

(a) Derive the cosmological redshift from the conservation law arising from the space translation symmetry of the flat Friedmann-Lemaître Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \quad (3)$$

(b) Illustrate with a causal diagram the notion of a horizon in cosmology. Derive an expression for the physical distance $d_{hor}(t)$ to the horizon as a function of time t and evaluate this in a flat FLRW universe that has always been matter dominated. In this universe, compute the current age t_0 and the present horizon size $d_{hor}(t)$ in terms of the Hubble constant H_0.

3. Gravitational (GW) waves (5 pts)

Consider the spacetime

$$ds^2 = -dt^2 + (1 + f(t-z))\,dx^2 + (1 - f(t-z))\,dy^2 + dz^2 \quad (4)$$

describing a plane gravitational wave propagating in the z-direction.

(a) Explain why it is impossible to detect a GW with a single test mass.

~~Consider two~~

(b) Consider two test masses, one at the origin and the other at (X, Y, Z) in the Cartesian coordinates used in (4). Derive the change in distance between the masses produced by the GW

(c) Sketch the GW waveform of the Nobel prize winning LIGO observation of GW and identify three key features of the waveform that can be used to qualitatively understand the nature of the source of this GW pattern. Briefly explain what can be learned from each of these.