GR Session 1: Special Relativity and Tensor Calculus

Friday October 5, 2012

1. Chapter 1 problem 12 of Carroll

2. (a) Steve Prefontaine moves with four-velocity $u^\mu = dx^\mu/d\tau$ with respect to a fixed coordinate system set up in an olympic stadium. What are the components of u^μ as a function of his three-velocity $\vec{u}$?

   (b) Derive the general expression for the Lorentz transformation which boosts from Steve's rest frame to the stadium's frame where his four-velocity is given by your answer to 2a. (*Hint: use the fact that the Lorentz transformation matrix is symmetric and should go to the identity in the limit* $\vec{u} \to 0$.)

   (c) Steve is actually running a race against Usain Bolt, who quickly passes him. Steve measures that Usain has passed him at a three-velocity $\vec{v}$. What is Usain Bolt's three-velocity vector $\vec{v}'$ in the frame of the stadium? Write your answer in terms of $\vec{v}_\parallel$ and $\vec{v}_\perp$ the components of $\vec{v}$ parallel resp. perpendicular to $\vec{u}$.

   (d) If $\vec{u}$ and $\vec{v}$ point in the same direction ($\vec{v}_\parallel = \vec{v}$ as is usually the case in a race) then the addition of velocities formula derived in 2c takes a special form. By replacing $|\vec{v}| \equiv \tanh\phi_v$ and $|\vec{u}| \equiv \tanh\phi_u$ show that ϕ_u and ϕ_v add in the normal sense. This makes it clear that addition of velocities in one spatial dimension can be viewed as a hyperbolic rotation in a plane. (*Hint: this is a simple application of a hyperbolic trigonometric identity.*)

   (e) Jonathan Borlée, Kevin Borlée and Jente Bouckaert are joined by $n-3$ other Belgian runners for a relay race. At the hand-off, Kevin moves away from Jonathan with speed v in Jonathan's frame. When Kevin hands off to Jente, he sees Jente again move away with speed v in his frame. This happens n times, after each handoff of the baton. What is the velocity of the nth runner in the frame of the olympic stadium? What does this tend to in the limit as $n \to \infty$? (*Hint:* $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$)

3. Chapter 1 problem 2 of Carroll

4. Rocket propulsion works by conservation of momentum, in order to start moving from rest a rocket must launch its mass behind it. A rocket of mass M_r starts at rest on Earth and instantaneously starts to eject mass from its thruster. The ejected mass always shoots out at a speed u as measured by observers on the rocket. If the rocket reaches a final speed v as measured from the Earth, what do the Earthans measure as the final mass of the rocket M_f?

5. A particle of mass m and charge q is traveling through an electromagnetic field. In the language of special relativity we write the four-vector potential $A^\mu = (\phi, \vec{A})$ where ϕ is the electrostatic potential and $\vec{A}$ is the magnetic vector potential. The action of such a particle is given by

$$S = -m\int d\tau + q\int A_\mu dx^\mu \ , \tag{1}$$

   where $d\tau = \sqrt{-ds^2}$. Use this action to derive the relativistic equations of motion for a particle in an electromagnetic field. Some useful identities to use are:

$$\nabla\left(\vec{E}\cdot\vec{F}\right) = \vec{E}\times\left(\nabla\times\vec{F}\right) + \vec{F}\times\left(\nabla\times\vec{E}\right) + \left(\vec{E}\cdot\nabla\right)\vec{F} + \left(\vec{F}\cdot\nabla\right)\vec{E} \ ,$$

$$\frac{d\vec{C}}{dt} = \dot{\vec{x}}\cdot\nabla\vec{C} + \frac{\partial\vec{C}}{\partial t} \ .$$

   Difficult Bonus Question: Can you figure out how to take the $m \to 0$ limit of the above action and derive equations motion for a massless charged particle?