GR Session 2: Special Relativity, Tensor Calculus and Manifolds

Wednesday October 10, 2012

1. (D'Inverno 3.9) A particle moves from rest at the origin of a frame S along the x-axis with constant acceleration α (as measured in an instantaneous rest frame).

   (a) Show that the equation of motion is
   $$\alpha x^2 + 2c^2 x - \alpha c^2 t^2 = 0 \ .$$

   (b) Prove that light signals emitted after time $t = c/\alpha$ at the origin will never reach the receding particle.

   (c) A standard clock carried along with the particle is set to read zero at the beginning of the motion and reads τ at time t in S. Prove the following relationships:
   $$\frac{v}{c} = \tanh\frac{\alpha\tau}{c} \ , \quad \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \cosh\frac{\alpha\tau}{c} \ ,$$
   $$\frac{\alpha t}{c} = \sinh\frac{\alpha\tau}{c} \ , \qquad x = \frac{c^2}{\alpha}\left(\cosh\frac{\alpha\tau}{c} - 1\right) \ .$$

   (d) Show that, if $T \ll c^2/\alpha^2$, then during an elapsed time T in the inertial system, the particle clock will record approximately the time $T(1 - \alpha^2 T^2/6c^2)$. If $\alpha = 3g$, with $g = 9.8m/s^2$, find the difference in recorded times by the spaceship clock and those of the inertial system (i) after 1 hour, and (ii) after 10 days.

2. **Vectors** (D'Inverno 5.16) In $\mathbb{R}^2$, let $x^a = (x, y)$ denote Cartesian coordinates and $x'^a = (r, \phi)$ denote plane polar coordinates

   (a) If the vector field X has components $X^a = (1, 0)$, then find X'^a .

   (b) The operator grad can be written in each coordinate system as
   $$\nabla f = \partial_x f\,\hat{x} + \partial_y f\,\hat{y} = \partial_r f\,\hat{r} + \frac{1}{r}\partial_\phi f\,\hat{\phi}$$
   where f is an arbitrary function and
   $$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \ , \quad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \ .$$
   Take the scalar product of ∇f with $\hat{x}$, $\hat{y}$, $\hat{r}$ and $\hat{\phi}$ in turn to find the relationships between the operators ∂_x , ∂_y, ∂_r, ∂_ϕ .

   (c) Express the vector field X as an operator in each coordinate system. Use part (b) to show these expressions are the same.

   (d) If $Y^a = (0, 1)$ and $Z^a = (-y, x)$, then find Y'^a, Z'^a and the operators Y, Z.

   (e) Evaluate all the Lie brackets of X, Y, and Z.

3. **Carroll chapter 2 problem 9** In Minkowski space, suppose $\star F = q\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$.

   (a) Evaluate $\mathrm{d}\star F = \star J$.

   (b) What is the two-form F equal to?

   (c) What are the electric and magnetic fields equal to for this solution?

(d) Evaluate $\int_V \mathrm{d} \star F$, where V is a ball of radius R in Euclidean three space at a fixed moment of time.

4. A particle of mass m and charge q is traveling through an electromagnetic field. In the language of special relativity we write the four-vector potential $A^\mu = (\phi, \vec{A})$ where ϕ is the electrostatic potential and $\vec{A}$ is the magnetic vector potential. The action of such a particle is given by

$$S = -m \int d\tau + q \int A_\mu dx^\mu \ , \tag{1}$$

where $d\tau = \sqrt{-ds^2}$. Use this action to derive the relativistic equations of motion for a particle in an electromagnetic field. Some useful identities to use are:

$$\nabla \left(\vec{E} \cdot \vec{F} \right) = \vec{E} \times \left(\nabla \times \vec{F} \right) + \vec{F} \times \left(\nabla \times \vec{E} \right) + \left(\vec{E} \cdot \nabla \right) \vec{F} + \left(\vec{F} \cdot \nabla \right) \vec{E} \ ,$$
$$\frac{d\vec{C}}{dt} = \dot{\vec{x}} \cdot \nabla \vec{C} + \frac{\partial \vec{C}}{\partial t} \ .$$

Difficult Bonus Question: Can you figure out how to take the $m \to 0$ limit of the above action and derive equations motion for a massless charged particle?