GR Sessions 3 & 4: Curvature

Wednesdays October 17 and 24, 2012

1. **Metric Compatibility**

   (a) Prove that the metric is covariantly constant, $\nabla_\mu g_{\rho\sigma} = 0$.

   (b) Deduce that you can raise and lower indices inside covariant derivatives, $g_{\mu\nu}\nabla_\sigma V^\nu = \nabla_\sigma V_\mu$.

   (c) Use (1a) to prove that the inverse metric $g^{\mu\nu}$ and the epsilon tensor $\epsilon_{\mu\nu\rho\sigma}$ (normalized so that $\epsilon_{0123} = \sqrt{|g|}$) are also covariantly constant.

2. **Carroll 3.5** Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \ .$$

   (a) Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latituted ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

   (b) Take a vector with components $V^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector as a function of θ?

3. Find all Killing vectors for a space with the following metric:

$$ds^2 = x^2\, dx^2 + x\, dy^2 \ .$$

4. **Properties of Killing Vectors**

   (a) Show that any Killing vector X satisfies

$$g^{\mu\nu}\nabla_\mu\nabla_\sigma X_\nu - R_{\sigma\mu}X^\mu = 0 \ .$$

   (b) Show that any Killing vector X satisfies

$$\nabla_\mu\nabla\nu X_\rho = R_{\rho\nu\mu\sigma}X^\sigma \ .$$

5. **Isometries**

   (a) Carroll Problem 3.14. Consider the three Killing vectors of the two-sphere:

$$\begin{aligned} R &= \partial_\phi \ , \\ S &= \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi \ , \\ T &= -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi \ . \end{aligned}$$

   Show that their commutators must satisfy the following algebra:

$$\begin{aligned} [R, S] &= T \\ [S, T] &= R \\ [T, R] &= S \ . \end{aligned}$$

   (b) What commutators will the Killing vectors of Minkowski space (in any number of dimensions) obey?

6. **Gravitational Time Dilation** Carroll 3.6

7. **Parallel Transport** Carroll 3.7