Exam Functional Analysis, 1 February 2024

Problem 1 Define on $H = L^2([-1,1])$ the map $T : H \to H$ defined by $(Tf)(x) = x^2 f(x)$.

a) Show that $T \in B(H)$

b) Show that T has no eigenvalues

c) Compute $\|T\|$

d) Prove that $\sigma(T) = [0,1]$. (Hint: If $t \in [0,1]$, note that $T - t^2 I$ does not surject on $g \in L^2[-1,1]$ defined by $g(x) = x - t$.)

Problem 2 Let X be a Banach space and consider its dual X^* equipped with the weak* topology. Let $x, y \in X$ and define $f : X^* \to \mathbb{R} : \omega \mapsto |\omega(x)| - |\omega(y)|$.

a) Show that f is continuous for the weak* topology

b) Prove that there exists a $\omega_0 \in B$ such that

$$f(\omega_0) = \sup_{\omega \in B} f(\omega).$$

Here, B is the closed unit ball in X^*

c) Show that $f(\omega) = 0$ for all $\omega \in B$ if and only if there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $y = \lambda x$

d) Write $Y = \{\omega \in X^* \mid \omega(x), \omega(y) \geq 0\}$ and let $K \subseteq Y$ be a nonempty convex compact set such that $f(\omega) \geq 1$ for all $\omega \in \operatorname{ext} K$. Show that $f(K) \subseteq [1, \infty)$

Problem 3 Let H be a Hilbert space and $P \in B(H)$ selfadjoint, satisfying $P^4 = P$.

a) Compute $\|P\|$

b) Show that P is an orthogonal projection (Hint: (not on the exam sheet!) Note that when $T \in B(H)$ is selfadjoint and $\sigma(T) = \{0\}$, then $T = 0$.)

Problem 4 Let H be a Hilbert space.

a) Assume $L, R : H \to H$ are linear maps satisfying $\langle Lx, y \rangle = \langle x, Ry \rangle$ for all $x, y \in H$. Show that L is bounded.

b) Let $T \in B(H)$ be compact and selfadjoint. If $\varepsilon > 0$, then show that

$$\{\lambda \in \sigma(T) \mid |\lambda| \geq \varepsilon\}$$

is a finite set.