Statistical Inference : Examen

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1 Theory

1.1 Question 1

Let $R(T_n, \theta)$ be risk function.

1. Define what is meant by an estimator.
2. Let X be a r.v. with density function $f(\cdot;\theta)$. Show that if we consider a continuous prior density then the Bayes estimator is given by

$$T_B = \frac{\int \theta \prod_{i=1}^{n} f(x_i;\theta)\pi(\theta)d\theta}{\int \prod_{i=1}^{n} f(x_i;\theta)\pi(\theta)d\theta}.$$

3. Now consider $X \sim Bernoulli(\theta)$ and suppose that the prior density is given by a $Beta(\alpha, \beta)$ distribution. Calculate the Bayes estimator.

1.2 Question 2

1. What is meant by a Neyman-Pearson test?
2. Define what is meant by a test of size α and show that there always exists such a test of size α.
3. Let $X \sim \mathcal{N}(\mu, 4)$. We wish to test

$$H_0 : \mu = -1 \text{ versus } H_1 : \mu = 1.$$

   (a) Consider a critical region of the form

$$R = \{\bar{X}_n > c\}.$$

      i. Determine c such that this test has size α.
      ii. Calculate the probability of a type II error.

   (b) Now consider

$$R = \{\sum_{i=1}^{n} iX_i > a\}.$$

      i. Determine a such that this has size α.
      ii. Calculate the probability of a type II error.
      iii. Compare the probabilities of type II error and comment.

1.3 Question 3

1. Describe the Wilcoxon signed-rank test in detail and comment on all assumptions made.
2. Assume that there are no ties and calculate the expected value of the estimator for this test.
3. Calculate the variance again with the assumption of no ties.
4. Describe how your calculations differ when you consider ties.
5. If you assume no ties calculate the exact distribution of T^+ for $n = 4$.

2 Exercises

2.1 Question 1

Consider X with cumulative distribution

$$F(x) = P(X \leq x) = (1 - e^{-x})^{\frac{1}{\theta}}$$

for $x \geq 0$ and $\theta > 0$.

1. Find the cumulative distribution of $W = -\ln(1 - e^{-X})$. What is its distribution?
2. Find the MLE for θ and derive an asymptotic normality result.
3. Find an approximate $100 \times (1 - \alpha)\%$ confidence interval. Limit the use of approximations.
4. Use the first part of this question to derive an exact confidence interval.
5. For $0 \leq \beta \leq 1$ and calculate the β-th quantile ξ_β. Propose an estimator for this quantile and quote an asymptotic normality result.

2.2 Question 2

We have two unknown masses θ_A and θ_B. We measure these with a machine with a measuring error that has expected value 0 and variance σ^2. We first measure A five times, then B five times and finally A and B five times.

1. Formulate a regression model. Explain all the notation you introduce. In the following questions explicitly state all the assumptions you have to make.
2. Find a least squared estimator for θ_A and θ_B.
3. Calculate the variances of these estimators.
4. Explain how you would test if the sum of masses equals 10g.
5. Explain how you would test if the masses were equal.
6. If we just measured the sum of the masses fifteen times wich of the previous questions would you not be able to answer.