Functional Analysis Fall 2022, Final exam: February 3, 2023

Problem 1. We denote by H the Hilbert space $L^2([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$. Define $T : H \to H$ by $(Tf)(x) = xf(x) + f(x)$.

(a) Show that T is a bounded linear operator.

(b) Show that T does not have eigenvalues.

(c) Compute the spectrum $\sigma(T)$ of T.

Problem 2.

(a) Let X be a vector space. We endow X with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Assume that there is $c > 0$ for which $\|x\|_2 \leq c\|x\|_1$, for any $x \in X$. Prove that there is $d > 0$ such that $\|x\|_1 \leq d\|x\|_2$, for any $x \in X$.

(b) Let X and Y be Banach spaces. Suppose that a linear map $T : X \to Y$ is continuous when X and Y are equipped with their weak topologies. Show that T is a bounded linear operator.

(c) Let X be a Banach space and $A \subset X$. Prove that A is bounded if and only if for all $\omega \in X^*$ we have that $\sup_{x \in A} |\omega(x)| < \infty$.

Problem 3. Let H be a Hilbert space and let $P, Q \in B(H)$ be non-zero orthogonal projections.

(a) Show that $\|P\| = 1$.

(b) Show that $\sigma(P - \frac{1}{2}I) \subset \{-\frac{1}{2}, \frac{1}{2}\}$.

(c) Show that $\|P - Q\| \leq 1$.

Problem 4.

(a) Show that the extreme points of the closed unit ball of a Hilbert space H equals to $\{x \in H \mid \|x\| = 1\}$.

(b) Find the extreme points of the closed unit ball of $\ell^1(\mathbb{N})$.