BA Physics **January 2023**

Introduction to General Relativity

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1. *Black Holes* (7 pts)
The Schwarzschild geometry is given by the metric (with $G = c = 1$),

$$ds^2 = -\left[1 - \frac{2M}{r}\right] dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2 d\Omega_2^2 \qquad (1)$$

where $d\Omega_2^2$ is the line element of S^2. The transformation from the coordinates (t, r, θ, ϕ) to Kruskal coordinates (U, V, θ, ϕ), defined for $r > 2M$ by

$$U = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh(t/4M), \quad V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh(t/4M)$$

yields the Schwarzschild geometry in the following form, valid for all $r > 0$,

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2 d\Omega_2^2 \qquad (2)$$

with $r(U, V)$ defined implicitly by the relation

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = U^2 - V^2. \qquad (3)$$

So-called Penrose coordinates (U', V') are defined as

$$V' - U' \equiv \tan^{-1}(V - U) \qquad V' + U' \equiv \tan^{-1}(V + U) \qquad (4)$$

(a) Draw a Kruskal diagram and indicate the singularity and the horizon, as well as the worldlines of an infalling and a distant observer. Show that in a Kruskal diagram $|dV/dU|$ must be greater than unity for any timelike particle worldline, even if the particle is moving non-radially.
(b) Explicitly carry out the transformation to Penrose coordinates (U', V')

2. *Gravitational Lensing* (6 pts)
(a) Describe as concretely and concisely as possible two astrophysical or cosmological examples of gravitational lenses and what we can learn from these.
(b) Derive the difference in path length, to first order in the angle β between the lens and the source, in terms of the Einstein angle θ_E and the basic parameters of the lens configuration.

3. *Gravitational Waves* (7 pts)
Consider a binary system consisting of two compact objects of equal mass M in orbit about each other in the (x, y) - plane under their mutual gravitational attraction. The orbit is circular with radius R and orbital frequency Ω.
(a) Derive the Newtonian relation between R and Ω.
(b) The gravitational wave (GW) luminosity is given by

$$L_{GW} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle \quad (5)$$

where the second mass moment is defined as

$$I^{ij}(t) = \int d^3x \mu(t, \mathbf{x}) x^i x^j \quad (6)$$

with μ the rest-mass density. Calculate the GW luminosity and express your result in terms of M and Ω.
(c) Gravitational waves carry away energy from the system. This causes the orbital frequency to change. Find a relation between Ω and $\dot{\Omega}$.
(d) Sketch the waveform of the Nobel prize winning LIGO observation of the burst of gravitational waves from two colliding objects and discuss how your results (a) - (c) can be used to qualitatively understand the nature of these objects, their radius and their mass(es), starting from the observed GW waveform.