Exam Numerical Simulation of Differential Equations

Saturday, September 3, 2022, Prof. Giovanni Samaey

- The exam is written. You may use the course materials, and handwritten notes, but no other books.
- Write clearly legible and structure your answer. Start a new page for each question.
- The exam takes a maximum of 3 hours.

Good luck!

Vraag 1. Consider 3 methods for the initial value problem $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ with initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$:

- **Method 1:**

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

- **Method 2:**

$$\boldsymbol{y}^{k+1} - \frac{18}{11}\boldsymbol{y}^{k} + \frac{9}{11}\boldsymbol{y}^{k-1} + \frac{2}{11}\boldsymbol{y}^{k-2} = \frac{6}{11}\Delta t \boldsymbol{f}(t^{k+1}, \boldsymbol{y}^{k+3}).$$

- **Method 3:**

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{y}^{k} + \frac{\Delta t}{2}\boldsymbol{f}\left(t^{k+1/2}, \boldsymbol{\xi}\right) \\ \boldsymbol{y}^{k+1} &= \boldsymbol{y}^{k} + \Delta t \boldsymbol{f}\left(t^{k+1/2}, \boldsymbol{\xi}\right). \end{aligned}$$

Now answer the following three questions:

1. Part I: classification (2 points)
   a) Write the first method in formula form.
   b) One of these methods contains a typo. Find the typo and explain how to notice it. (Simply stating that the method is different in the course notes is not sufficient ...). **In the rest of the question, we pretend that the typo has been corrected.**
   c) Describe and classify each of the methods. Be as clear and precise as possible. Give the order of each method.
   d) Which stability region belongs to each method? Make a sketch illustrating the essential characteristics of each stability region and briefly justify using the theoretical stability properties of the method.
2. Part II: behavior (2 points)
   a) What behavior do you expect for each of the methods for the system of linear equations

$$\begin{cases} y_1' &= y_2 \\ y_2' &= -y_1 \end{cases} \tag{1}$$

   if you keep increasing the time step size Δt? Explain your answer.

b) Consider for a moment the different possibilities for the behavior of a system of differential equations, and the corresponding properties we would like to see after time discretization. In our limited catalog here, do we have a method available for every possible case? If so, please note. If not, what kind of method is missing?

3. Part III: application (3 points)

   a) Consider now a system of equations of the form

   $$u_k' = \frac{1}{\Delta x^2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + u_k(1 - u_k), \qquad k = 1, \ldots, K.$$

   Which of the above time discretisation methods do you recommend for this problem? Why? When in doubt, give your reasoning and the steps you would take to decide which method is most suitable.

   b) If you choose an implicit method, how would you solve the nonlinear system at each time step? Consider iteration scheme, starting value, and stopping criterion.

   c) If your iterative method requires you to solve linear systems in each iteration: how would you solve those linear systems?

Question 2. Consider the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0\,,$$

with a being a real constant. The Lax-Friedrichs method is given as

$$u_j^{k+1} = \frac{u_{j+1}^k + u_{j-1}^k}{2} - a\Delta t\frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x}.$$

a) What order does this diagram have in space and time? Derive the order of consistency. (1 point)

b) Prove convergence using the maximum principle. (2 points)

c) Show that you can rewrite this method as a discretization of the advection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2},$$

with $\gamma = \Delta x^2/(2\Delta t)$. (1 point)

d) Derive the stability condition of the Lax-Friedrichs method, and explain from your knowledge of the course. Compare with the CFL condition. (1.5 points)

e) Compare the stability of the Lax-Friedrichs method with the method of central differences. Use your results from questions (c) and (d) to do so. (1.5 points)

Question 3. Briefly answer the following questions. Limit yourself to about half a page per answer. The goal is not to define all concepts in detail, but to use the necessary terms correctly in your answer. If a figure or formula can help, use it. Derivations, proofs, etc., are not necessary unless explicitly asked otherwise.

a) In the finite element method, one of the steps in the derivation is the transition to local coordinates in each element. What is the importance of this step for the finite element method? (What does this transition allow that would not otherwise be possible?) (1 point)

b) What is the main difference between error estimation in linear multistep methods and error estimation in Runge–Kutta methods? What is the reason for this difference? What is the effect on the computational cost? (2 points)

c) What is a characteristic of a hyperbolic conservation law? How do these characteristics look for a hyperbolic conservation law with constant, space-dependent, or solution-dependent coefficients. (2 points)

e) What is the maximal order of a *convergent* linear multistep method? What causes non-convergence of methods of higher order? (1 point)