Operator Algebras

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1. Let $\mathcal{X} = \{u_n \mid n \in \mathbb{N}\}$ and $\mathcal{R} = \{u_n^* u_n = u_n u_n^* = \mathbf{1}\}$.
   (a) (3 points) Justify rigorously that $B := \mathrm{C}^*\left(\mathcal{X} \mid \mathcal{R}\right)$ exists.
   (b) (7 points) Let A be a unital, separable C*-algebra. Show that there is an ideal $J \subseteq B$ such that B/J is $*$-isomorphic to A.

2. (10 points) Let A be a C*-algebra, $\alpha : A \to A$ a $*$-automorphism, and $\phi \in \mathcal{S}(A)$ a state with induced GNS-triple $(\pi, \mathcal{H}, \xi)$. Suppose $\phi = \phi \circ \alpha$.

   Show that there exists a unitary $\mathcal{H} \to \mathcal{H}$ such that $U\pi(a)U^* = \pi(\alpha(a))$ for all $a \in A$.

   Hint: Review the GNS-construction, in particular how π and $\mathcal{H}$ were defined. How would U need to look on the canonical dense subspace of $\mathcal{H}$?

3. (a) (7 points) Let $A \subseteq \mathcal{L}(\mathcal{H})$ be a concrete C*-algebra with $\mathbf{1} \in A$ and set $M := A''$. Show that for every unitary u in M, there exists a net of unitaries v_λ in A, such that $v_\lambda \to u$ in the strong operator topology.

   Hint: Remember that $\exp(it) = \cos(t) + i\sin(t)$ for all $t \in \mathbb{R}$.

   (b) (3 points) Does the same hold for projections, i.e., for every projection p in M, there exists a net of projections q_λ in A, such that $q_\lambda \xrightarrow{\text{SOT}} p$. Give a sketch of proof or a motivated counterexample.

4. Let A be a non-zero C*-algebra. A state $\phi \in \mathcal{S}(A)$ is called faithful if for every non-zero x, $\phi(x^*x) \neq 0$.
   (a) (5 points) Show that if A is separable, there exists a faithful state on A.
   (b) (5 points) Give an example of a non-zero C*-algebra that has no faithful state.

   Hint: Think of commutative C*-algebras with very large Gelfand spectrum.