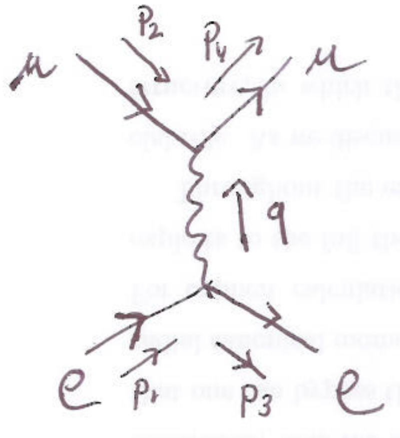
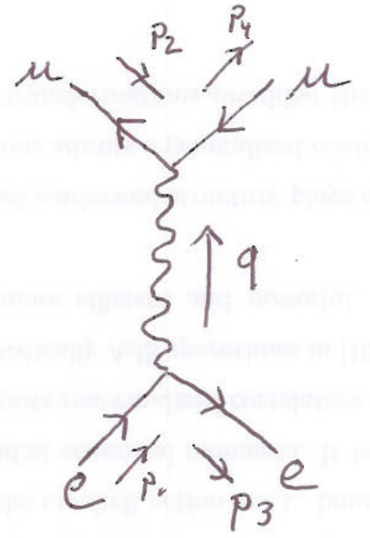
(1)

$e^- \mu^-$ scattering



$e^- \mu^+$ scattering.



$$e^-\mu^- : \quad \mathcal{M}_1 = -\frac{g_e^2}{(p_1-p_3)^2}\,[\bar u(3)\gamma^\mu u(1)][\bar u(4)\gamma_\mu u(2)]$$

$$e^-\mu^+ : \quad \mathcal{M}_2 = -\frac{g_e^2}{(p_1-p_3)^2}\,[\bar u(3)\gamma^\mu u(1)][\bar v(2)\gamma_\mu v(4)].$$

- Casimir on $\mathcal{M}_1$ $\to$ see 7.126 in griffiths (of course you cannot write such a thing on your exam)

- Casimir on $\mathcal{M}_2$ $\to$ Follow analogy & redo analysis.

the first factor $[\bar u(3)\gamma^\mu u(1)]$ is the same as for $\mathcal{M}_1$. To understand whether there is any difference between process 1 and 2 we need to compute

$$G = \sum_{\text{spins}} [\bar v(2)\gamma_\mu v(4)][\bar v(2)\gamma_\nu v(4)]^*$$

notice that

$$G = \sum_{2,4} \left[\bar{V}(2)\, \gamma_\mu\, V(4)\right]\left[\bar{V}(4)\, \gamma_\nu\, V(2)\right]$$

$$= \sum_{2} \bar{V}(2)\, \gamma_\mu \left(\sum_4 V(4)\, \bar{V}(4)\right) \gamma_\nu\, V(2)$$

$$= \sum_{2} \bar{V}(2)\, \gamma_\mu (\not{p}_4 - M)\, \gamma_\nu\, V(2)$$

$$= \sum_{2,i,j} \bar{V}(2)_i \left[\gamma_\mu (\not{p}_4 - M)\, \gamma_\nu\right]_{ij} V(2)_j$$

$$= \sum_{ij} \left(\sum_2 V(2)_j\, \bar{V}(2)_i\right) \left[\gamma_\mu (\not{p}_4 - M)\, \gamma_\nu\right]_{ij}$$

$$= \sum_{ij} \left[\not{p}_2 - M\right]_{ji} \left[\gamma_\mu (\not{p}_4 - M)\, \gamma_\nu\right]_{ij}$$

$$= \mathrm{Tr}\left[(\not{p}_2 - M)\, \gamma_\mu (\not{p}_4 - M)\, \gamma_\nu\right]$$

So we have

$$\langle |M_1|^2 \rangle = \frac{g_e^4}{4(p_1 - p_3)^4} \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + mc)\, \gamma^\nu (\not{p}_3 + mc)\right] \times \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + Mc)\, \gamma_\nu (\not{p}_4 + Mc)\right]$$

$$\langle |M_2|^2 \rangle = \frac{g_e^4}{4(p_1 - p_3)^4} \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + mc)\, \gamma^\nu (\not{p}_3 + mc)\right] \times \mathrm{Tr}\left[\gamma^*_\nu (\not{p}_2 - M)\, \gamma^*_\mu (p_4 - Mc)\right].$$

m = mass e

M = mass μ

③

2 differences between $\langle |M_1|^2 \rangle$ and $\langle |M_2|^2 \rangle$:

→ $M \leftrightarrow -M$

→ $\mu \leftrightarrow \nu$ (indices)

or equivalently $2 \leftrightarrow 4$.

Now we demonstrate that these differences do not matter.

1) First we show that $M \to -M$ in $\langle |M_1|^2 \rangle$ or $\langle |M_2|^2 \rangle$ has no effect. Note that the amplitudes are quadratic in M:

$\langle |M_{2,1}|^2 \rangle \sim a M^2 + bM + c M$. Then we show that $b = 0$. Hence $\langle |M|^2 \rangle$ is same for $M \to -M$.

To see that $b=0$, just expand second factor.

$$\mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M)\right] =$$

$$\mathrm{Tr}\left[\gamma_\mu \gamma_\nu\right] M^2 + \underbrace{\left(\mathrm{Tr}\left[\gamma_\mu \not{p}_2 \gamma_\nu\right] + \mathrm{Tr}\left[\gamma_\mu \gamma_\nu \not{p}_4\right]\right)}_{=0} M$$

$$+ \mathrm{Tr}\left[\gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4\right]$$

since is trace of odd numbers of γ-matrices

The two remaining terms are

- $\mathrm{Tr}[\gamma_\mu \gamma_\nu] M^2 \rightsquigarrow$ obviously symmetric in $\mu \leftrightarrow \nu$

- $\mathrm{Tr}[\gamma_\mu \not{p}_2 \gamma_\nu \not{p}_4]$ - Use trace theorem.

$$= \mathrm{Tr}[\gamma_\mu \gamma_\sigma \gamma_\nu \gamma_\rho] p_2^\sigma p_4^\rho$$

$$= 4(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\sigma\rho} + g_{\mu\rho} g_{\sigma\nu}) p_2^\sigma p_4^\rho$$

$$= 4((p_2)_\mu (p_4)_\nu - g_{\mu\nu} (p_2 \cdot p_4) + (p_2)_\nu (p_4)_\mu)$$

$\Rightarrow$ Symmetric in $\mu \leftrightarrow \nu$.

Trace theorem

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma] =$$

$$\mathrm{Tr}[(-\gamma^{\nu\mu} + 2g^{\mu\nu}) \gamma^\lambda \gamma^\sigma] = -\mathrm{Tr}[\gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\sigma] + 2g^{\mu\nu} \mathrm{Tr}[\gamma^{\lambda\sigma}] = +\mathrm{Tr}[\gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\sigma]$$

$$-2g^{\mu\lambda} \mathrm{Tr}[\gamma^\nu \gamma^\sigma] + 8g^{\mu\nu} g^{\lambda\sigma} = -\mathrm{Tr}[\gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu] + 2g^{\mu\sigma} \mathrm{Tr}[\gamma^\nu \gamma^\lambda]$$

$$-8g^{\mu\lambda} g^{\nu\sigma} + 8g^{\mu\nu} g^{\lambda\sigma}$$

Now use ~~...~~ cyclicity of trace on first term to change it to $-\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma]$. This equals minus the left-hand side. Hence.

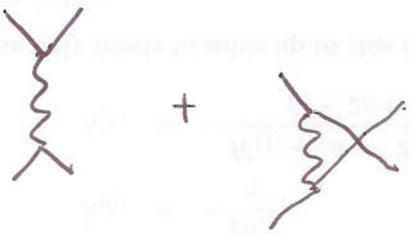
$$2\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma] = 8g^{\mu\sigma} g^{\nu\lambda} - 8g^{\mu\lambda} g^{\nu\sigma} + 8g^{\mu\nu} g^{\lambda\sigma}.$$

(difficult question. ~~[illegible]~~
↳ little credits) (5)

$e^- e^-$ vs $e^- e^+$ scattering

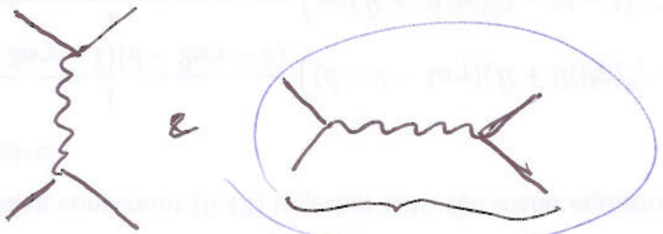
→ This is different. Quick argument

$e^- e^-$:



$S = 1/2$.

$e^- e^+$



physically different proces

=> Should be different (= ok answer)

=> In non-rel limit doesn't contribute.
Then might become the same (= also ok answer)

Question 2 (6)

a) If $\mathcal{L}$ is real, $\mathcal{L} = \mathcal{L}^*$. Let us check.

$$\mathcal{L}^* = \mathcal{L}^c = -i\,\overline{\partial_\mu \Psi}\,\overline{\gamma^\mu}\,\overline{\overline{\Psi}} - m\overline{\Psi}\,\overline{\overline{\Psi}}$$

$$= -i\,\overline{\partial_\mu \Psi}\,\gamma^\mu\,\Psi - m\overline{\Psi}\Psi$$

$$= -i\partial_\mu[\overline{\Psi}\gamma^\mu\Psi] + \underbrace{i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi}_{=\mathcal{L}}$$

So $\mathcal{L}$ is not real. But the imaginary part is a total derivative. According to Euler-Lagrange, total derivatives do not affect the equations of motion.

b) $\partial_\mu\Psi \to \mathcal{D}_\mu\Psi = [\partial_\mu + iqA_\mu]\Psi$.

If $\Psi \to e^{i\theta}\Psi$ & $A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\theta$.

the action is gauge invariant. For completeness one needs to add the kinetic term for A_μ:

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi.$$

c) $J^\mu \sim \overline{\Psi}\gamma^\mu\Psi$. $\partial_\mu J^\mu = 0$ (See first exercise class with Ellen)

$$\partial_\mu J^\mu = (\partial_\mu\overline{\Psi})\gamma^\mu\Psi + \overline{\Psi}\gamma^\mu\partial_\mu\Psi = im\overline{\Psi}\Psi - im\overline{\Psi}\Psi = 0$$