General Relativity – Orientation Test

November 2013

1 Show that the commutator $[\mathrm{X}, \mathrm{Y}]$ of two vector fields X and Y is itself a vector field. Derive an explicit expression for its components $[\mathrm{X}, \mathrm{Y}]^\mu$.

2 Show that the Christoffel connection satisfies

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|} \tag{1}$$

where $|g|$ is the absolute value of the determinant g of the metric.

3 (a) Show that the Weyl tensor satisfies a version of the Bianchi identity:

$$\nabla^\rho C_{\rho\sigma\mu\nu} = \nabla_{[\mu} R_{\nu]\sigma} + \frac{1}{6} g_{\sigma[\mu} \nabla_{\nu]} R$$

(b) Show that any Killing vector satisfies

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu$$

4 Consider the metric

$$ds^2 = -(du dv + dv du) + a^2(u) dx^2 + b^2(u) dy^2 \tag{2}$$

where a and b are unspecified functions.
(a) Calculate the Christoffel symbols and Riemann tensor for this metric.
(b) Use Einstein's equations in vacuum to derive equations obeyed by $a(u)$ and $b(u)$.
(c) Show that an exact solution can be found, in which both a and b are determined in terms of an arbitrary function $f(u)$.

5 Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{3}$$

(a) Show that lines of constant longitude are geodesics, and that the only line of constant latitude that is a geodesic is the equator.
(b) Take a vector with components $V^\mu = (1, 0)$ and parallel-transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?

6 Show that if we decompose the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then the components of the Riemann tensor are

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}) \tag{4}$$

to linear order in $h_{\mu\nu}$. Show explicitly that this linearized Riemann tensor is invariant under the gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{5}$$