1 Special Relativity

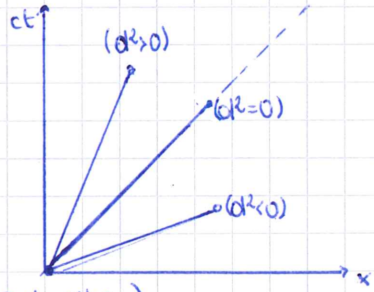
Minkowski Spacetime

classical mechanics: Cartesian coord syst on $\mathbb{R}^3$ (x, y, z) with
$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$
relativity: 4D Minkowski space-time with coord x^μ ; $\mu = 0, 1, 2, 3$
$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$
→ invariant distance between two events $d^2 = (x_1^0 - x_2^0)^2 - (x_1^1 - x_2^1)^2 - (x_1^2 - x_2^2)^2 - (x_1^3 - x_2^3)^2$ (mostly minus conv.)

if $d^2 > 0$: timelike seperation
- in causal contact

$d^2 = 0$: lightlike seperation
- can send a light ray between the points

$d^2 < 0$: spacelike seperation
- not in causal contact (cannot communicate with each other)



simplify: $x_2^\mu = 0$, $x_1^\mu = x^\mu$
→ Einstein notation ↱ greek index repeated twice (once upstairs and once downstairs) you sum over all values of the index
$$d^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$
$$= \eta_{\mu\nu} x^\mu x^\nu$$
$$= \Sigma_{\mu\nu} \eta_{\mu\nu} x^\mu x^\nu$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
Minkowski metric

Lorentz transformations (Λ)

= linear transformations of Mink. coords x^μ that preserve distances d^2
$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad d^2 = \eta_{\mu\nu} x^\mu x^\nu \longmapsto (d')^2 = \eta_{\mu\nu} x'^\mu x'^\nu$$
$$= \eta_{\mu\nu} (\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma)$$

since $d^2 = (d')^2 \Rightarrow \Lambda^T \eta \Lambda = \eta$ (Lorentz condition)

show this for any x, y: ← $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma$
$$x^T \eta y = (\Lambda x)^T \eta (\Lambda y) = x^T \Lambda^T \eta \Lambda y$$
$$\Rightarrow \eta = \Lambda^T \eta \Lambda$$

- metric tensor (bilinear form) $\eta_{\mu\nu}$ can be used to lower or raise indices
$$x_\mu = \eta_{\mu\nu} x^\nu = \Sigma_\nu \eta_{\mu\nu} x^\nu \qquad x^\nu = (x^0, x^1, x^2, x^3)$$
$$\rightarrow d^2 = \eta_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu = x_\nu x^\nu \rightarrow \text{Lorentz scalar}$$
(dummy indices)

$\eta_{\mu\nu} = \eta^{-1}_{\mu\nu} = \eta^{\mu\nu}$

↳ $\eta_{\mu\nu} = \eta^{\mu\nu} \rightsquigarrow \eta^{-1} = \eta \rightarrow \eta^\nu{}_\mu = \delta^\nu{}_\mu$

check this $\eta_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\rho} \eta^{\sigma\rho} = \begin{cases} \eta_{\nu\rho} \eta^{\mu\rho} & (\mu = 0) \\ -\eta_{\nu\rho} \eta^{\mu\rho} & (\mu \neq 0) \end{cases} = \eta^{\mu\nu}$

- inner product: $v \cdot w = v^\mu w^\nu \eta_{\mu\nu} = v^0 w^0 - \vec{v} \cdot \vec{w}$ → invar. under Lorentz transfos
(upstairs downstairs)
- under Lorentz transfo
$$v'^\mu = \Lambda^\mu{}_\nu v^\nu \qquad v^\mu \text{ is a } (1,0) \text{- or contravariant (tensor) vector}$$
$$w'_\mu = \Lambda_\mu{}^\nu w_\nu \qquad w_\mu \text{ is a } (0,1) \text{- or covariant (tensor) vector}$$
→ where $\Lambda_\mu{}^\nu = \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^\rho{}_\sigma$

⇒ a general (m,n)-tensor transforms as follows
$$T'^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = \Lambda^{\mu_1}{}_{\rho_1} \dots \Lambda^{\mu_m}{}_{\rho_m} \Lambda_{\nu_1}{}^{\sigma_1} \dots \Lambda_{\nu_n}{}^{\sigma_n} T^{\rho_1 \dots \rho_m}{}_{\sigma_1 \dots \sigma_n}$$

check that for contravariant v^μ, $v_\mu = \eta_{\mu\nu} v^\nu$ transforms as a covariant vector
$$v'_\mu = \eta_{\mu\nu} v'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\rho v^\rho = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} v_\sigma = \eta_{\mu\nu} \eta^{\rho\sigma} \Lambda^\nu{}_\rho v_\sigma$$
$$= \eta_{\mu\nu} \eta^{\sigma\rho} \Lambda^\nu{}_\rho v_\sigma$$
$$= \Lambda_\mu{}^\sigma v_\sigma$$

__show that__ $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are invariant [(0,2) and (2,0) tensors]

$$\eta_{\mu\nu} = \Lambda_\mu{}^\sigma \eta_{\sigma\rho} \Lambda_\nu{}^\rho \qquad \text{since } \eta = \Lambda^T \eta \Lambda$$
$$= \eta'_{\mu\nu}$$
$$\eta^{\mu\nu} = \Lambda^\mu{}_\sigma \eta^{\sigma\rho} \Lambda^\nu{}_\rho = \eta'^{\mu\nu}$$

the trace $T^\mu{}_\mu$ of $T^\mu{}_\rho$ is invariant [(1,1) tensor]

$$\Lambda_\nu{}^\mu = (\Lambda^{-1})^\mu{}_\nu \rightarrow \Lambda^\nu{}_\sigma (\Lambda^\nu{}_\rho)^{-1} = \eta_{\sigma\nu}\eta^{\nu\rho} = \eta^\rho{}_\sigma$$

$$T'^\mu{}_\mu = \Lambda^\mu{}_\rho \Lambda_\mu{}^\sigma T^\rho{}_\sigma = \eta^\rho{}_\sigma T^\rho{}_\sigma = \delta^\rho{}_\sigma T^\rho{}_\sigma = T^\rho{}_\rho$$

the contraction of the first upper index and first lower index makes from (p,q) tensor $T^\mu{}_\rho$ a (p-1,q-1) tensor

$$T^{\mu_1\mu_2\dots\mu_p}{}_{\rho_1\rho_2\dots\rho_q} \longrightarrow \eta^{\mu_1}{}_{\rho_1} T^{\mu_1\mu_2\dots\mu_p}{}_{\rho_1\rho_2\dots\rho_q} = T^{\alpha\mu_2\dots\mu_p}{}_{\alpha\rho_2\dots\rho_p}$$
$$= \delta \quad T \qquad = T^{\mu_2\dots\mu_p}{}_{\rho_2\dots\rho_p}$$

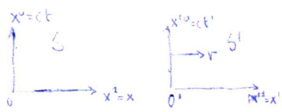
if the indices on a (p,q) tensor are lowered and raised such that now p' indices are upstairs and q' downstairs (with p+q = p'+q') then this new object is a (p',q') tensor [consistency Einstein notation]

$$T^{\mu_1\dots\mu_p}{}_{\rho_1\dots\rho_q} \longrightarrow T^{\mu_1\dots\mu_{p'}}{}_{\rho_1\dots\rho_{q'}}$$

L

Lorentz boosts

boost coordinate frame $S \mapsto S'$ in x^1-direction : $\underset{S'}{x'^\mu} = \Lambda^\mu{}_\nu \underset{S}{x^\nu}$



$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = v/c$$

$$\leadsto \begin{cases} t' = \gamma(t - \frac{v}{c^2}x) & (1) \\ x' = \gamma(x - vt) & (2) \end{cases} \quad \text{or} \quad \begin{cases} t = \gamma(t' + \frac{v}{c^2}x') & (3) \\ x = \gamma(x' + vt') & (4) \end{cases}$$

__show that__ this is a Lorentz transfo and write down the matrices corresponding to boosts in the x^2 and x^3 direction

x^1: $x' = \Lambda x = \begin{pmatrix} \gamma x^0 - \gamma\beta x^1 \\ -\gamma\beta x^0 + \gamma x^1 \\ x^2 \\ x^3 \end{pmatrix} \rightarrow (s')^2 = \eta_{\mu\nu} x'^\mu x'^\nu = (\gamma x^0 - \gamma\beta x^1)^2 - (-\gamma\beta x^0 + \gamma x^1)^2 - (x^2)^2 - (x^3)^2$

$$= \gamma^2(1-\beta^2)[(x^0)^2 - (x^1)^2] - (x^2)^2 - (x^3)^2$$
$$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \eta_{\mu\nu}x^\mu x^\nu$$

x^2: $\Lambda' = A^{-1}\Lambda A$

$$A = (x^1 \leftrightarrow x^2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Bigg\} \quad \Lambda' = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

x^3: $\Lambda'' = B^{-1}\Lambda B$

$$B = (x^1 \leftrightarrow x^3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Bigg\} \quad \Lambda'' = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

__what is__ the matrix corresponding to a boost in the $x^2 + x^3$ direction?

$$\Lambda''' = R^{-1}_{x^2+x^3 \leftrightarrow x^1} \Lambda R_{x^2+x^3 \leftrightarrow x^1} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$$

$$\text{of } = R^{-1}_{x^2+x^3 \leftrightarrow x^2} \Lambda' R_{x^2+x^3 \leftrightarrow x^2} \xrightarrow[\theta = 45°]{\text{rotatie}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

effects relativity

* time dilation

  clock at origin S' $\rightarrow x' = 0$

  $\overset{(3)}{\Rightarrow} \Delta t = \gamma \Delta t'$, $\Delta t \geq \Delta t'$ since $\gamma \geq 1$

  clock goes slower in S' frame (1 second in S' passes as γ seconds in S)

* length contraction

  rod from $x' = 0$ to $x' = L$ in S'

  $\overset{(2)}{\Rightarrow}$ at $t = 0$ in S: $L' = \gamma L \rightarrow L = \gamma^{-1} L'$, $L \leq L'$

  (a single instant in time)

* addition of velocities

  in S': $x' = w't'$ $\longrightarrow$ in S: $w = \frac{x}{t} = \frac{w' + v}{1 + vw'/c^2}$

  $t = \gamma(1 + \frac{vw'}{c^2})t'$
  $x = \gamma(w' + v)t'$

Relativistic mechanics (motion)

- remember: in a closed system energy and momentum are conserved
  this should hold in any inertial frame
- time
  time τ on the moving clock is an invariant time coordinate
  → τ = proper time → time particle in rest res (S')
  → $c^2 \Delta\tau^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$
  explain why we cannot use this formulation for particles travelling at the speed of light
  here $\frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} = c^2$
  $\Rightarrow \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = c^2\Delta t^2 - c^2 \Delta t^2 = 0$
- velocity
  4-velocity $\eta^\mu = \frac{dx^\mu}{d\tau}$ (observer frame / particle clock) with $\vec{\eta} = \frac{d\vec{x}}{d\tau}$ the proper velocity

  explain why this is a 4-vector
  τ is invariant under transfos & x^μ is a 4-vect ⇒ η^μ will transform as a 4-vect
  $\eta'^\mu = dx'^\mu/d\tau = \Lambda^\mu{}_\sigma \, dx^\sigma/d\tau = \Lambda^\mu{}_\sigma \eta^\sigma$
  is dx^μ/dt a 4-vector?
  no, since t isn't invariant under transfos → norm won't be invariant
  for any observer, the distance that P has traveled
  $d^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$
  for P (who finds himself to be at rest) this is: ($\vec{x}_p = 0$) $d^2 = c^2 \Delta\tau^2$
  → 4-velocity of moving P: $\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma c, \gamma \frac{d\vec{x}}{dt}) = (\gamma c, \gamma \vec{v})$
  since $d\tau = \gamma^{-1} dt$ (time dilation)
  the Minkowski norm is $\eta^\mu \eta_\mu = c^2$
  show this $\|\eta^\mu\| = \eta \cdot \eta = \eta_\mu \eta^\nu = \eta_{\mu\nu} \eta^\mu \eta^\nu$ → using def γ
  or $c^2 \Delta\tau^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$
  $\Rightarrow c^2 = \eta_{\mu\nu} \frac{\Delta x^\mu}{\Delta\tau} \frac{\Delta x^\nu}{\Delta\tau} = \eta_{\mu\nu} \eta^\mu \eta^\nu$

- momentum & energy
  4-momentum $p^\mu = m\eta^\mu = (\gamma mc, \gamma m\vec{v}) = (\gamma mc, \vec{p})$
  norm $p^\mu p_\mu = m^2c^2$
  since p^0: $p^0 = \gamma mc$
  $p^0 c = \gamma mc^2 \approx mc^2 + \frac{1}{2} mv^2 + \frac{3}{8} mc^{-2} v^4 + \dots$ [expand for $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, small] $v \ll c$
  verify this $p^0 c = mc^2 \frac{1}{\sqrt{1 - v^2/c^2}} \approx mc^2 \left[\underbrace{1}_{f(0)} + \underbrace{\frac{1}{2}\frac{v^2}{c^2}}_{f'(0)x} + \underbrace{\frac{3}{8}\frac{v^4}{c^4}}_{\frac{1}{2!}f''(0)x^2} \right]$
  since for $f(x) = (1-x)^{-1/2}$: $f'(x) = \frac{1}{2}(1-x)^{-3/2}$, $f''(x) = \frac{3}{4}(1-x)^{-5/2}$
  $p^0 c$ = rest energy + kin. energy + rel. corrections
  ⇒ $E \equiv p^0 c = \gamma mc^2$ relativistic energy

@ zero velocity $E = mc^2$

  thus $p^\mu = (E/c, \vec{p})$
  ⇒ $p^\mu p_\mu = m^2c^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$
  kinetic energy particle: $T = E - mc^2 = mc^2(\gamma - 1)$
  massless particles: $E = |\vec{p}|c$ → photon: $E = \hbar\nu$
- conservation 4-momentum
  spec. rel: particles not subject to an external force follow a geodesic curve;
  and in Minkowski a geodesic fulfills $\frac{d}{d\tau} \eta^\mu = 0$ (straight line) → massive particle? → light time like
  → conservation 4-mom.

- conservation 4-momentum
  - $p_A^\mu + p_B^\mu = p_A'^\mu + p_B'^\mu$

explain why if valid in one inertial frame, valid in all.
if valid in one, we can just transform to another inertial frame and we'll get the corresponding equal.

→ implies conservation total energy and 3-momentum (this eq.)

→ distinction elastic ($m=cte$) and non-elastic collisions (sticky: $m<0$, explosive: $m>0$)

- 2 frames: laboratory → target at rest
  CM → total 3-momentum is zero

explain why CM-frame always exists
because . . .

- exercises on collisions (extra page)

Maxwell theory

- Maxwell equations

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho & (1) \\ \vec{\nabla}\cdot\vec{B} = 0 & (2) \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0 & (3) \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{j} & (4) \end{cases} \longrightarrow F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

antisymm (2,0) tensor

with ρ = charge density
$\vec{j}$ = current density

and define $j^\mu = (c\rho, \vec{j})$

↳ continuity eq: $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$ ⟶ $\partial_\mu j^\mu = 0$

Check this $\partial_\mu j^\mu = \frac{\partial(c\rho)}{\partial ct} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ ok!

why does this expresses conservation of charge?

$$Q = \int_V \rho\, dV \rightarrow \frac{dQ}{dt} = \int_V \frac{\partial\rho}{\partial t} dV = -\int_V \vec{\nabla}\cdot\vec{j}\, dV = -\int_{\partial V} \vec{j}\cdot d\vec{A}$$

net change charge within volume
= net current into the volume } charge conservation

↳ Maxwell equations /w source on RHS [(1) & (4)]

4 eq. $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$

check this $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$: $\frac{4\pi}{c} j^0 = 4\pi\rho = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \partial_\mu F^{\mu 0}$

$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi\vec{j}}{c}$: $\frac{4\pi}{c} j^1 = \frac{4\pi}{c} j_x = (\nabla\times\vec{B})_x - \frac{1}{c}\frac{\partial E_x}{\partial t}$
$= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial ct}$
$= \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \partial_\mu F^{\mu 1}$

show that this eq. leads to the cont. eq

$\partial_\nu j^\nu = \frac{c}{4\pi}\partial_\nu\partial_\mu F^{\mu\nu} = \frac{c}{8\pi}(\partial_\mu\partial_\nu F^{\mu\nu} + \partial_\nu\partial_\mu F^{\nu\mu})$ (dummy var)
$= \frac{c}{8\pi}(\partial_\mu\partial_\nu F^{\mu\nu} + \partial_\mu\partial_\nu F^{\nu\mu})$ $F^{\mu\nu} = -F^{\nu\mu}$ since antisymm.
$= 0$

remaining Maxwell equations

4 eq. $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$

$\epsilon^{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_2} F_{\mu_3\mu_4} = 0$ curl in 4D /w ϵ the Levi-Civita symbol

$\epsilon^{0123} = -1$ $\epsilon^{0213} = 1$ → { $\mu_1\mu_2\mu_3\mu_4$ even perm (0,1,2,3) → −1; $\mu_1\mu_2\mu_3\mu_4$ odd perm → +1; rest → 0 }

check this
$\mu_1 = 0$: $\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} - \partial_1 F_{32} - \partial_2 F_{13} - \partial_3 F_{21} = 0$
antisymm eig $F_{\mu\nu}$
→ $\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = 0$
$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla\cdot\vec{B} = 0$

$\mu_1 = i$: $\partial_0 F_{jk} + \partial_j F_{0k} + \partial_k F_{j0} = 0$
$\frac{\partial B_i}{\partial t} + \frac{\partial E_k}{\partial j} - \frac{\partial E_j}{\partial k} = \frac{\partial B_i}{\partial t} + (\nabla\times E)_i = 0$

- Maxwell equations

Is ε a tensor? no

if ε would be a tensor: $\varepsilon_{\mu_1\mu_2\mu_3\mu_4} = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\Lambda_{\mu_3}{}^{\nu_3}\Lambda_{\mu_4}{}^{\nu_4}\,\varepsilon_{\nu_1\nu_2\nu_3\nu_4}$

but $\varepsilon_{0132} \neq \varepsilon_{0123}$ → so not a tensor

$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\mu$, $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_1}F^{\mu_2\mu_3} = 0$] =

or $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

<u>check this</u>

$\partial_\rho F_{\mu\nu} = \partial_\rho\partial_\mu A_\nu - \partial_\rho\partial_\nu A_\mu$

$= \partial_\mu\partial_\rho A_\nu - \partial_\mu\partial_\nu A_\rho + \partial_\nu\partial_\mu A_\rho - \partial_\nu\partial_\rho A_\mu$

$= \partial_\mu F_{\rho\nu} + \partial_\nu F_{\mu\rho}$

$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_1}F_{\mu_2\mu_3} = \varepsilon^{\mu_1\mu_2\mu_3\mu_4}[\partial_{\mu_2}F_{\mu_1\mu_3} + \partial_{\mu_3}F_{\mu_2\mu_1}]$

$= -\varepsilon^{\mu_2\mu_1\mu_3\mu_4}\partial_{\mu_2}F_{\mu_1\mu_3} - \varepsilon^{\mu_3\mu_2\mu_1\mu_4}\partial_{\mu_3}F_{\mu_2\mu_1}$

$= +2\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_1}F_{\mu_2\mu_3}$

$\Rightarrow \varepsilon\,\partial_{\mu_1}F_{\mu_2\mu_3} = 0$

<u>check that</u> $\partial_\mu X$ with X some scalar, a one-form is

$\partial_\mu' = \Lambda_\mu{}^\rho\partial_\rho \quad \rightarrow (\partial_\mu X)' = \Lambda_\mu{}^\rho\partial_\rho X$

$\partial^\mu = \eta^{\mu\nu}\partial_\nu$ where $\partial_\nu = \partial/\partial x^\nu$

$A^\mu = (\phi, \vec{A})$ with ϕ the scalar pot, $\vec{A}$ the vector pot.

write $\vec{E}, \vec{B}$ in terms of $\vec{A}$ and ϕ

$E_1 = \partial^1 A^0 - \partial^0 A^1 = -\partial\phi/\partial x - 1/c\,\partial\vec{A}_x/\partial t$

$B_1 = \partial^3 A^2 - \partial^2 A^3 = -\partial\vec{A}_y/\partial z + \partial\vec{A}_z/\partial y = (\nabla\times\vec{A})_x$

$\Rightarrow \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

the Maxwellvect A^μ doesn't describe $\vec{E}, \vec{B}$ uniquely

→ gauge invariance

$A'_\mu = A_\mu + \partial_\mu\lambda$ leaves $F_{\mu\nu}$ (x Maxwell eq.) invariant

⇒ Lorentz gauge : $\partial_\mu A^\mu = 0$

<u>verify</u> $F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$

$= \partial^\mu A^\nu - \partial^\nu A^\mu + \underbrace{\partial^\mu\partial^\nu\lambda - \partial^\nu\partial^\mu\lambda}_{=0}$

$= F^{\mu\nu}$

the Maxwell equations simplify to

$\Box A^\mu = \frac{4\pi}{c} j^\mu$, $\Box = \partial^\mu\partial_\mu$ (d'Alembertian)

Lorentz gauge does not uniquely remove gauge redundancy

any λ for which $\Box\lambda = 0$ can still be used $\partial_\mu A^\mu = 0 = \partial_\mu(A^\mu + \partial^\mu\lambda)$

⇒ further useful, but non-covariant gauge:

Coulomb gauge $A^0 = \phi = 0$ (in absence of any source, $j^\mu = 0$)

<u>check this</u>

$A^\mu + \partial^\mu\lambda = (0$, andere dingen$)$ ⇒ $\phi = -\partial^t\lambda$ → $\Box\lambda = 0$?

- Quantum mechanical interpretation

solve free wave eq. $\partial_\mu A^\mu = 0$ x search mom-eigenstates

* use $p_\mu = i\hbar\partial_\mu$

derived from $\vec{p} = -i\hbar\vec{\nabla}$ $E = i\hbar\frac{\partial}{\partial t}$

<u>check this</u> $\mu = 0$: $p_0 = E/c = i\hbar\,\partial/\partial(ct) = i\hbar\partial_0$

$\mu \neq 0$: $p_\mu = -p^\mu = i\hbar\partial_\mu$

→ solutions of the form

$A^\nu = a\exp(-\frac{i}{\hbar}p_\mu x^\mu)\,\varepsilon^\nu(p)$

will be energy and mom. eigenstates and requires $p^2 = 0$ (foton)

$k^2 = 0$ (alg)] $k_\mu \equiv p_\mu$

$\Box A' = 0$ x $\partial_\mu\partial^\mu A^\nu \sim p_\mu p^\mu A^\nu$

<u>why?</u>

use Coulomb
x Lorentz gauge
$A^\mu = \begin{pmatrix}0\\ \vec{A}\end{pmatrix}$, $\partial_\mu A^\mu = 0$
no charge
$\partial_\mu F^{\mu\nu} = 0$
$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$
$\Box A^\mu = 0$
(golfvgl)
→ $\Box\vec{A} = 0$ plane waves
ontbinden in Fourier basis
→ oplossing: $A' = \varepsilon'\exp(\frac{-i}{\hbar}k_\mu x^\mu)$

- Quantummechanical interpretation / helicity

$A^\nu = a\exp(-\frac{i}{\hbar}p_\mu x^\mu)\varepsilon^\nu(p)$, $\partial_\mu A^\mu = 0$, $A^0 = 0$

$\Rightarrow \vec{p}\cdot\vec{\varepsilon} = 0$, $\varepsilon^0 = 0$ with $\vec{\varepsilon}$ polarizationvect.

→ only 2 independent polarisation states, orthogonal to direction of motion

Thus a massless field/particle has 2 helicity states,

eigenstates of the spin operator

without Coulomb gauge:

$p^\mu \varepsilon_\mu^{(1,2)} = 0$ → chosen orth. & norm: $(\varepsilon_\mu^{(1)})^* \varepsilon_\mu^{(2)} = 0$, $(\varepsilon^\mu)^*\varepsilon_\mu = -1$

with Coulomb gauge:

$$\sum_{s=1,2} \varepsilon_i^{(s)}(\varepsilon_j^{(s)})^* = \delta_{ij} - \hat{p}_i\hat{p}_j \qquad \text{(completeness relation)}$$

2 rdintoest 2 onafh opl

→ 6 equations

<u>prove this</u>

LHS & RHS are 2 matrices, only equal if LHS v = RHS v

choose basis: $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}, \hat{p}_k$

then $\sum_k[\sum_s \varepsilon_i^{(s)}\varepsilon_k^{(s)}]\varepsilon_k^{(1)} = \sum_s \varepsilon_i^{(s)} \sum \varepsilon_k^{(s)}\varepsilon_k^{(1)} = \varepsilon_i^{(1)}\sum_k(\varepsilon_k^{(1)})^2$

$= \varepsilon_i^{(1)}$ $\quad |\vec{\varepsilon}| = 1$

and $\sum_k[\delta_{ik} - \hat{p}_i\cdot\hat{p}_k]\varepsilon_k^{(1)}$

$= \sum_k \delta_{ik}\varepsilon_k^{(1)} - \sum_k \hat{p}_i[\hat{p}_k\varepsilon_k^{(1)}]$

$= \varepsilon_i^{(1)} - 0$

↳ valid for $\varepsilon^{(1)}$, analog for $\varepsilon^{(2)}$ and $\hat{p}$

- Lagrangian

<u>explain</u> why a Hamiltonian isn't Lorentz invariant

I don't know

minimums $\partial S = 0$

action: $S = \int dx^0 dx^1 dx^2 dx^3\, \mathcal{L}(\phi^i, \partial\phi^i)$ for Lagrangian density $\mathcal{L}$ depending on fields ϕ^i

↳ → EL: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi^i]}\right) = \frac{\partial\mathcal{L}}{\partial\phi^i}$ (eq. of motion – Lorentzinvar. if L a scalar)

$= \int d^4x\, \partial\mathcal{L}$

$= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi^i}\partial\phi^i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\partial\partial_\mu\phi^i\right]$

$= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi^i}\partial\phi^i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\partial_\mu\partial\phi^i\right]$

$= -\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\partial\phi^i\right)$

$\rightarrow \frac{\partial\mathcal{L}}{\partial\phi^i} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}$

for free Maxwell theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad\rightarrow\quad \frac{\partial\mathcal{L}}{\partial\partial_\rho A_\sigma} = -\frac{1}{2}\frac{\partial F_{\mu\nu}}{\partial\partial_\rho A_\sigma}F^{\mu\nu}$$

$$= -\frac{1}{2}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu)F^{\mu\nu}$$

$$= -F^{\rho\sigma}$$

<u>check this</u> $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\sigma)} = -\frac{1}{4}\left[\eta^{\rho\alpha}\eta^{\sigma\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \eta^{\sigma\alpha}\eta^{\rho\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + \eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \eta^{\mu\sigma}\eta^{\nu\rho}(\partial_\mu A_\nu - \partial_\nu A_\mu)\right]$$

$$= -\frac{1}{4}[F^{\rho\sigma} - F^{\sigma\rho} + F^{\rho\sigma} - F^{\sigma\rho}] = -\frac{1}{4}[F^{\rho\sigma} + F^{\rho\sigma} + F^{\rho\sigma} + F^{\rho\sigma}] = -F^{\rho\sigma}$$

for $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4\pi}{c}A_\mu j^\mu$ (see below)

$\frac{\partial\mathcal{L}}{\partial A_\sigma} = -\frac{4\pi}{c}j^\sigma$

$\longrightarrow \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\mu$

add source J^μ to the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4\pi}{c}A_\mu j^\mu$$

$$\Rightarrow \partial_\rho F^{\rho\sigma} = \frac{4\pi}{c}j^\sigma$$

continuity eq. source ($\partial_\mu j^\mu = 0$) implies gauge invariance of the action

- Lagrangian

check that a gauge transformation indeed gives an extra total derivative
and that a total derivative added to a lagr. density leaves the EL equations unchanged

* A gauge transfo leaves $F_{\mu\nu}$ unchanged, and we have $A_\mu \to A_\mu + \partial_\mu \lambda$

Thus $\mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{4\pi}{c} A_\mu j^\mu - \frac{4\pi}{c} (\partial_\mu \lambda) j^\mu$

$$= \mathcal{L} - \frac{4\pi}{c} \partial_\mu (\lambda j^\mu) + \frac{4\pi}{c} \lambda \underbrace{\partial_\mu j^\mu}_{=0} \quad \text{(cont eq.)}$$

$$= \mathcal{L} - \frac{4\pi}{c} \partial_\mu (\lambda j^\mu)$$

→ extra derivative

* Consider lagrangian density $\mathcal{L}(\phi^i, \partial\phi^i)$ and denote $\mathcal{L}' = \mathcal{L} + \partial_\mu f^\mu$

Assuming that $f^\mu = 0$ on the boundary, we now have

$$S' = \int_\Omega d^4x \, (\mathcal{L} + \partial_\mu f^\mu) = S + \int_\Omega d^4x \, \partial_\mu f^\mu$$

$$= S + \underbrace{\int dV f^\mu}_{=0}$$

$=0$ since boundary condition
or something w/ a constant that doesn't add to extremizing

variatie
Lagr. niet gevonden

$$\int \nabla \cdot d\vec{F} = \int_{\partial\Omega} d\vec{F} \cdot d\vec{S} \; ?$$

↳ no variation, only on boundary?

Excercises on collisions

1] $\pi \rightarrow \mu + \nu$ what is the speed of the muon?
@rest

$\vec{p}_\pi = p_\mu + p_\nu \rightarrow p_\nu = p_\mu - p_\pi \rightarrow p_\nu^2 = p_\mu^2 - 2p_\mu p_\pi + p_\pi^2$

$p_\mu p_\pi = p_\mu^0 p_\pi^0 = m_\pi c\, p_\mu^0 = m_\pi E_\mu$ (enkel p^0; $\vec{p}=0$ aangezien $v_\pi = 0$) $\rightarrow 0 = m_\mu^2 c^2 + m_\pi^2 c^2 - 2 m_\pi E_\mu$ *

$p_\mu = p_\pi - p_\nu \rightarrow p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu + p_\nu^2$
$m_\mu^2 c^2 = m_\pi^2 c^2 - 2 m_\pi E_\nu$

$\Rightarrow 2 m_\pi E_\nu = (m_\pi^2 - m_\mu^2) c^2$

$\rightarrow |\vec{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c$) $E_\nu = |\vec{p}_\nu| c = |\vec{p}_\mu| c$ since $\vec{p}_\mu = -\vec{p}_\nu$

relativistic: $\vec{p} = \gamma m \vec{v}$, $E = \gamma m c^2$; $\vec{v} = \frac{\vec{p} c^2}{E_\mu}$

$\rightarrow |\vec{v}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c \cdot \frac{c^2}{E_\mu}$

$= \frac{m_\pi^2 - m_\mu^2}{m_\mu^2 + m_\pi^2} c$) *

2] $p + p \rightarrow p + p + p + \bar{p}$
threshold kinetic energy lab frame for incident proton?
$p \rightarrow \cdot p$ ↳ in CM-frame: all produced part. at rest
CM: $(p_T')^\mu = (4mc, 0, 0, 0)$
lab frame: $p_T^\mu = (mc, 0, 0, 0) + (E/c, |\vec{p}|, 0, 0)$ — (int energy; moving along x-direction) — $(p^\mu)^2$ is frame independent

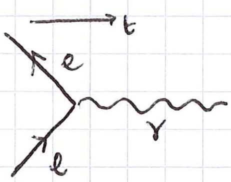
$\rightarrow (p_T^\mu)^2 = 16 m^2c^2 = (E/c + mc)^2 - |\vec{p}|^2$
$\Rightarrow 16 m^2 c^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 + m^2c^2 + 2Em$
$= m^2c^2 + m^2c^2 + 2Em$
$\Rightarrow E = 7mc^2$

3] $\underset{m}{o} \longrightarrow \quad \longleftarrow \underset{m}{o}$ with each T' in CM
what is their relative kin. energy? (i.e. in the rest frame of one of the part)

CM: $(p_T^\mu)' = (2E'/c, 0, 0, 0)$
lab: $p_T^\mu = (E/c + mc, \vec{p})$ (1 part moving, 1 part at rest)

↳ squaring both: $4E'^2/c^2 = E^2/c^2 - |\vec{p}|^2 + m^2c^2 + 2Em$
$= 2m^2c^2 + 2Em$
$\Rightarrow 2E'^2 = mc^2(E + mc^2)$
$T = E - mc^2$; $2(T' + mc^2)^2 = mc^2(2mc^2 + T)$
$T' = E' - mc^2$; $\Rightarrow T = \frac{2(T' + mc^2)^2}{mc^2} - 2mc^2$
$= 4T'\left(1 + \frac{T'}{2mc^2}\right)$

4]



Demonstrate why $\gamma \rightarrow e^- + e^+$ cannot occur
$e^- + e^+ \rightarrow \gamma$
In CM: $p_T' = (2mc, 0, 0, 0)$
after collision this should be the same
but impossible for a single photon
since p_T' doesn't square to 0
→ photon has no rest frame

total 3 mom = 0
but then 3mom photon 0
→ no energy
→ no photon

2 Relativistic Quantum Mechanics

probability currents in quantum mechanics

- probability density $\rho(x,y,z,t)\,dx\,dy\,dz$ = prob. to find a part. in vol. el. $dx\,dy\,dz$ around point (x,y,z) at time t

QM: $\rho(x,y,z,t) = \Psi^* \Psi$

if no decay: total probability is constant   unitarity: hamilton w/ no lost particles

→ cont. eq: $\vec{\nabla}\cdot\vec{j} + \partial\rho/\partial t = 0$ (w $\vec{j}$ = prob. current density)

→ static volume V: $\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \vec{\nabla}\cdot\vec{j}\, dV = -\int_{\partial V} \vec{j}\cdot d\vec{S}$

$$\rightarrow \vec{j} = \frac{\hbar}{2mi}\left[\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right]$$

using SE: $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$

with for 1 particle $H = \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V$ (operator) (hermitian)

which is never Lorentz invariant since time ~ 1st deriv (LHS), space ~ 2nd deriv (RHS)

for Ψ^* the SE is $-i\hbar\frac{\partial\Psi^*}{\partial t} = H\Psi^*$

check the expr. for the prob. current density

$$i\hbar\vec{\nabla}\cdot\vec{j} = -i\hbar\,\partial\rho/\partial t = -i\hbar\left[\partial\Psi^*/\partial t\;\Psi + \partial\Psi/\partial t\;\Psi^*\right]$$
$$= -\left[\left(\hbar^2/2m\,\nabla^2\Psi^* - V\Psi^*\right)\Psi + \left(-\hbar^2/2m\,\nabla^2\Psi + V\Psi\right)\Psi^*\right]$$
$$= -\hbar^2/2m\,\nabla^2\Psi^*\,\Psi + V\Psi^*\Psi + \hbar^2/2m\,\nabla^2\Psi\cdot\Psi^* - V\Psi\Psi^*$$
$$= \hbar^2/2m\left(-\Psi\nabla^2\Psi^* + \Psi^*\nabla^2\Psi\right)$$
$$= \hbar^2/2m\,\nabla\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

normalizable wave functions ($\int_V |\Psi|^2 = 1$) die off sufficiently fast at infinity

$\int_{\partial V_\infty}\vec{j}\cdot d\vec{S} = 0 \;\rightarrow\; \frac{\partial}{\partial t}\int_V \rho\, dV = 0$ total prob. always conserved

non-relativistic quantum mechanics

- SE: $i\hbar\,\partial_t\Psi = \left[-\hbar/2m\,\nabla^2 + V\right]\Psi$

never Lorentz invariant (solutions ↦ solutions) (reasoning above)

- Galilean transfo

$$\begin{cases}\vec{x}' = \vec{x} - \vec{v}t\\ t' = t\end{cases} \rightarrow \begin{cases}\Psi(\vec{x},t) = \Psi(\vec{x}'+\vec{v}t',t')\\ \partial_t\Psi(\vec{x},t) = \partial_{t'}\Psi(\vec{x}'+\vec{v}t',t') - \vec{v}\cdot\vec{\nabla}_{x'}\Psi(\vec{x}'+\vec{v}t',t')\\ \vec{\nabla}_x\Psi(\vec{x},t) = \vec{\nabla}_{x'}\Psi(\vec{x}'+\vec{v}t',t')\end{cases}$$

since effect of linear mixing

$$x'^\mu = \sum_\rho M^\mu{}_\rho x^\rho \Rightarrow \partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\nu}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \sum_\rho \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial}{\partial x'^\rho} = \sum_\rho M^\rho{}_\mu \frac{\partial}{\partial x'^\rho}$$
$$= \sum_\rho M^T{}_\mu{}^\rho \frac{\partial}{\partial x'^\rho}$$

rows, columns

$$\Rightarrow \partial_\mu = M^T{}_\mu{}^\rho\,\partial'_\rho$$

thus for
$$\begin{pmatrix}t'\\x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}1&0&0&0\\-v_x&1&0&0\\-v_y&0&1&0\\-v_z&0&0&1\end{pmatrix}\begin{pmatrix}t\\x\\y\\z\end{pmatrix} \Rightarrow \begin{pmatrix}\partial_t\\\partial_x\\\partial_y\\\partial_z\end{pmatrix} = \begin{pmatrix}1&-v_x&-v_y&-v_z\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\partial_{t'}\\\partial_{x'}\\\partial_{y'}\\\partial_{z'}\end{pmatrix}$$

$$\Rightarrow \partial_t = \partial_{t'} - \vec{v}\cdot\vec{\nabla}_{x'} \qquad \vec{\nabla}_x = \vec{\nabla}_{x'}$$

applying to SE to find new wave equation

$$i\hbar\left[\partial_{t'} - \vec{v}\cdot\vec{\nabla}_{x'}\right]\Psi(\vec{x}'+\vec{v}t',t') = \left[-\hbar^2/2m\,\vec{\nabla}^2_{x'} + V(\vec{x}'+\vec{v}t',t')\right]\Psi(\vec{x}'+\vec{v}t',t')$$

redefine $\Psi(\vec{x}'+\vec{v}t',t') = \Psi'(\vec{x}',t')\exp\left(i/\hbar\; m\vec{v}\cdot\vec{x}' + i/\hbar\; mv^2/2\; t'\right)$

$$\Rightarrow i\hbar\,\partial_{t'}\Psi'(\vec{x}',t') = \left[-\hbar^2/2m\,\vec{\nabla}^2 + V'(\vec{x}',t')\right]\Psi'(\vec{x}',t')$$

with $V'(\vec{x}',t') = V(\vec{x}-\vec{v}t,t')$

thus: in absence of a potential V (and V'), Ψ' satisfies the same eq. as Ψ'

→ Galilean invariance

for Galilean covariance V does not need to vanish

- Galilean transfo

$i\hbar\,\partial_{t'}\Psi'(\vec{r}\,',t') = [-\hbar^2/2m\,\vec{\nabla}^2 + V'(\vec{r}\,',t')]\,\Psi'(\vec{r}\,',t')$

<u>what</u> is the physical reason V has to vanish for Gal. invariance?

Galilean invariance → independent of frame/speed

thus $V'(\vec{r},t) = V(\vec{r}-\vec{v}t,t)$ $\quad [\vec{r}\,'=\vec{r}-\vec{v}t,\ t'=t]$

this means that the potentials have the same value everywhere and become physically irrelevant

<u>could</u> we have predicted the unitary transfo?

unitary transfo $\Psi(\vec{r}\,'+\vec{v}t',t') = \Psi'(\vec{r}\,',t')\exp[i/\hbar\,\underbrace{m\vec{v}\cdot\vec{r}}_{\text{direction}} + i/\hbar\,\underbrace{\tfrac{mv^2}{2}t'}_{\text{speed}}]$

phase factor

the energy is not conserved for a boost since it uses the time in its transfo)

→ energy difference = $\frac{mv^2}{2\hbar}t' \longleftarrow \exp(-iEt/\hbar)$

momentum also not conserved

→ momentum difference = $\frac{m\vec{v}\cdot\vec{r}}{\hbar} \longleftarrow \exp(-\frac{i}{\hbar}\vec{p}\cdot\vec{r})$

↳ transfo fixes this

relativistic wave equation: Klein-Gordon field

use operator correspondence

$E \to i\hbar\frac{\partial}{\partial t} \qquad \vec{p} \to -i\hbar\vec{\nabla} \qquad \&\ p_\mu = i\hbar\partial_\mu$

on rel. energy-momentum relation $(E/c)^2 - \vec{p}^{\,2} = m^2c^2$

⇒ Klein-Gordon equation

$\hbar^2\Box\Phi = -m^2c^2\Phi$ $\quad$ (2nd deriv time & space)

Wave operator (d'Alembertian)

$\Box = \partial_\mu\partial_\nu\eta^{\mu\nu} = \eta^{\mu\nu}\partial_\mu\partial_\nu$

$\vec{p} = p^i = -p_i$

<u>check this</u>

$[(E/c)^2 - \vec{p}^{\,2}]\Phi = m^2c^2\Phi$

$\Rightarrow -m^2c^2\Phi = -(i\hbar)^2[\partial^2/\partial(ct)^2 - \vec{\nabla}^2]\Phi$

$= \hbar^2[\frac{\partial^2}{\partial(ct)^2} - \vec{\nabla}^2]\Phi = \hbar^2\Box\Phi \quad (= p^2\Phi)$

$\hbar^2\Box\Phi$

$= \hbar^2\Box(\Phi' e^{-i/\hbar\, mc^2t})$

$= \hbar^2\Box\Phi'\, e^{-i/\hbar\, mc^2t}$

$+ \hbar^2\Phi'\,\Box e^{-i/\hbar\, mc^2t}$

$\hbar^2\Box\Phi' e^{\dots}$

$+ m^2c^4\Phi' e^{\dots}$

$= m^2c^2\Phi' e^{\dots}$

idle

apply on $\Phi' = \Phi\exp[i/\hbar\, mc^2t]$ ↘ logic: normally $\Psi \sim e^{-iEt/\hbar}$ → subtract the rest energy

↳ $-\hbar^2\frac{\partial^2\Phi'}{\partial t^2} + 2i\hbar mc^2\frac{\partial\Phi'}{\partial t} = -\hbar^2c^2\nabla^2\Phi'$

→ $-\frac{\hbar^2}{c^2}\partial_t^2\Phi' + 2i\hbar m\,\partial_t\Phi' = -\hbar^2\nabla^2\Phi'$

now non-rel. limit: $c \to \infty$ ⟶ $\frac{\partial^2\Phi'}{\partial t^2}$ very small wrt c^2, especially wrt $2i\hbar m$ and $-\hbar^2$

⇒ $i\hbar\,\partial_t\Phi' = -\frac{\hbar^2}{2m}\nabla^2\Phi'$ $\quad$ (Free wave S.E.)

if we define real quantities

$P = \frac{i\hbar}{2mc^2}\left(\Phi^*\frac{\partial\Phi}{\partial t} - \Phi\frac{\partial\Phi^*}{\partial t}\right) \qquad \vec{j} = \frac{\hbar}{2mi}(\Phi^*\vec{\nabla}\Phi - \Phi\vec{\nabla}\Phi^*)$

they obey following continuity eq

$\frac{\partial P}{\partial t} + \vec{\nabla}\vec{j} = 0$

<u>Check this</u> $\quad \frac{\partial P}{\partial t} + \vec{\nabla}\vec{j} = \frac{i\hbar}{2mc^2}\left(\Phi^*\frac{\partial^2\Phi}{\partial t^2} - \Phi\frac{\partial^2\Phi^*}{\partial t^2}\right) + \frac{\hbar}{2mi}(\Phi^*\vec{\nabla}^2\Phi - \Phi\vec{\nabla}^2\Phi^*)$

$= \frac{i\hbar}{2m}\left[\Phi^*\left(\frac{\partial^2\Phi}{\partial(ct)^2} - \nabla^2\Phi\right) + \Phi\left(-\frac{\partial^2\Phi^*}{\partial(ct)^2} + \vec{\nabla}^2\Phi^*\right)\right]$

$= \frac{i\hbar}{2m}[\Phi^*\Box\Phi - \Phi\Box\Phi^*] = \frac{i}{2m\hbar}[-m^2c^2\Phi^*\Phi + m^2c^2\Phi\Phi^*] = 0$

<u>show that</u> in the non-rel. limit we obtain the known expr. for prob. density and current

I don't know what to do

↳ P and $\vec{j}$ are the relativistic prob. dens. and current

However P is not all manifestly positive

and the plane wave solutions $\Phi_{E,\vec{k}} = A\exp(i\vec{k}\cdot\vec{r} - i\omega t)$

gives a positive and negative value for the energy

↳ disp. rel: $\hbar\omega^2 = m^2c^4 + \hbar^2c^2k^2$

with $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$

but $\hbar\omega = \pm\sqrt{\dots}$

(QFT solves the problem of $\pm|E|$ making sure that everything is stable and nothing decays to $-\infty$)

quantum field theory

particles created and destroyed – energy transfer to/from rest mass
→ wave eq. describes multiple particle states at once
el. waves ~ fields particles ~ field quanta
neg. energy states → pos. energies antiparticles

spin of a field / particle

- KG-field Φ: no internal structure → no spin
  spin 0 particles = scalars : invariant under Lorentz transfo
- spin ½ particles = spinors (rel. → Dirac spinors)
- QM: spin s particle /w wave function Ψ
  under spatial rot: $\Psi' = U_{\hat{n}}(\alpha)\Psi$ angle α over axis $\hat{n}$
  with $U_{\hat{n}}(\alpha) = \exp[-i/\hbar \; \alpha \hat{n}\cdot\vec{J}]$
  with $\vec{J}$ total angular momentum
  show that this is indeed a unitary operation ($UU^+ = U^+U = \mathbb{1}$ → $U^+ = U^{-1}$)
  $U_{\hat{n}}(\alpha)^+ = \exp[-i/\hbar \; \alpha\, \hat{n}\cdot\vec{J}]^+$
  $= \exp[i/\hbar \; \alpha\, \hat{n}\cdot\vec{J}]$ ↳ $\hat{n}\cdot\vec{J}$ = just a number since $\hat{n}, \vec{J}$ regular vectors
  $= U_{\hat{n}}(\alpha)^{-1}$

$[L, H] \neq 0$
$[J, H] = 0$
→ J is a conserved quantity
(operator: $J^+ = J$)

  where $\vec{J} = \vec{L} + \vec{S}$
  with $\vec{L}$ = orbit angular momentum $= \vec{r}\times\vec{p} = \vec{r}\times(i\hbar\vec{\partial})$
  and $\vec{S}$ = spin " "
  $\begin{cases} \vec{S}\times\vec{S} = i\hbar\vec{S} \\ \vec{S}\cdot\vec{S}\Psi = s(s+1)\hbar^2\Psi \end{cases}$
  ↳ spin 0 particles have $S=0$ ⇒ $\Psi'(\vec{r}\,',t) = \Psi(\vec{r},t)$
  check this $\Psi'(\vec{r},t) = U_{\hat{n}}(\alpha)\Psi(\vec{r},t) = \exp[-i/\hbar\;\alpha\,\hat{n}\cdot\vec{J}]\Psi(\vec{r},t)$
  $= \exp[-i/\hbar\;\alpha\,\hat{n}\cdot\vec{L}]\Psi(\vec{r},t) = \Psi(\vec{r}\,',t)$
  $\vec{L} = \vec{r}\times\vec{p}$

quantum states can be build from basis of eigstates of a comp. of $\vec{S}$
$S_z\Psi_{s,m_s} = m_s\hbar\Psi_{s,m_s}$, $m_s = -s, -s+1, \dots, s-1, s$
→ $2s+1$ eigenstates → $2s+1$ dim.
→ $\Psi_s = \sum_{m_s=-s}^{s} \Psi_{m_s}(\vec{r},t)\,\chi_{s,m_s}$ where $\chi_{s,m_s} = (0 \dots 0\; \underset{@m_s}{1}\; 0 \dots 0)^T$

⇒ spin 1 : 3-vect that rotates
spin ½ : spinors rotating via $\Psi' = \exp(\frac{i\theta}{\hbar} S_z)\Psi$

- RQM : $\vec{A} \to A^\mu$
  ⇒ spinor → Dirac spinor with 4 components
  describes 2 part. states (part & antipart)
  scalar, e.g. BEH-boson

Dirac equation

wanting to find 1st deriv Lorentz cov. wave eq. instead of KG-eq ($\Box\Phi = -\frac{m^2c^2}{\hbar^2}\Phi$)
→ rewrite $p^\mu p_\mu - m^2c^2 = 0$ (1)
(factorize) $= (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$ → would give 1st order $\gamma^\lambda p_\lambda - mc = 0$
⇒ $\beta^\kappa\gamma^\lambda p_\kappa p_\lambda + (\beta^\kappa p_\kappa - \gamma^\lambda p_\lambda)mc + m^2c^2 = 0$
to get (1) we need that $\beta = \gamma$
and $\gamma^\kappa\gamma^\lambda = \eta^{\kappa\lambda}$
however, for ordinary numbers this doesn't hold
show this say $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}$ then $\gamma^\mu\gamma^\nu = 0$ for $\mu \neq \nu$
now if the γ's are ordinary numbers, then $\gamma^\mu = 0$ or $\gamma^\nu = 0$ for $\mu \neq \nu$
but then $\gamma^\mu = 0$ for all μ
which is in contradiction with $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}$

Dirac equation

$\beta^\kappa \gamma^\lambda p_\kappa p_\lambda + (\beta^\kappa p_\kappa - \gamma^\lambda p_\lambda) mc - m^2c^2 = 0$

$\rightarrow \beta = \gamma$ and we want $\gamma^\mu \gamma^\nu p_\mu p_\nu = p^2$

for matrices $(\gamma^\mu)_{\alpha\beta}$
$(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)_{\beta\gamma}\, p_\mu p_\nu = p^2 \mathbb{1}_{\alpha\gamma}$
but matrices don't commute

$\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $\Longleftarrow$ anticommutation

show this $\frac{1}{2}\gamma^\mu\gamma^\nu (p_\mu p_\nu + p_\nu p_\mu) = p^2$

dummy

$\Leftrightarrow \frac{1}{2}\gamma^\mu\gamma^\nu p_\mu p_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu p_\mu p_\nu = p^2$

$\longrightarrow \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} p_\mu p_\nu = p^2$

$\Rightarrow$ $i\hbar\gamma^\mu \partial_\mu \Psi - mc\Psi = 0$

Dirac equation

- now in matrices:

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ $\rightarrow$ smallest dim. possible: 4x4

a well known basis:

check that this basis works
using commutations pauli matrices, you should show that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

σ^i: Pauli matrices
$\mathbb{1}_{2\times 2}$, $0_{2\times 2}$

Griffiths basis

$\rightarrow \gamma^\mu \rightsquigarrow (\gamma^\mu)_{\alpha\beta}$

Dirac wave has 4 components

giving $\sum_\beta [\,i\hbar(\gamma^\mu)_{\alpha\beta}\partial_\mu - mc\,\mathbb{1}_{\alpha\beta}]\,\Psi_\beta = 0$ for Dirac eq.

e.g. $\begin{pmatrix} |\uparrow\rangle e^- \\ |\downarrow\rangle e^- \\ |\uparrow\rangle e^+ \\ |\downarrow\rangle e^+ \end{pmatrix}$

with α, β: spinor indices
μ, ν: spacetime indices

each 4 indices

. explain why the Griffiths basis is not unique

for $i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0$

let K be any basis transfo, then $i\hbar K\gamma^\mu K^{-1}\partial_\mu K\Psi - mcK\Psi = 0$

now we denote $\Psi' = K\Psi$ and $\gamma'^\mu = K\gamma^\mu K^{-1}$

then these also satisfy Dirac equation

alternative basis:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

(only γ^0 differs)

Weyl basis

change of basis:

$\gamma^\mu_W = K\gamma^\mu_G K^{-1}$, $K = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$

$\hookrightarrow$ transformation K only works on spinor indices α
since γ act on the spinors, γ^μ_W acts on spinors as well

$\Rightarrow$ $\Psi_W = K\Psi_G$

- Dirac spinors

$\vec{J} = \vec{L} + \vec{S}$

and $H = c\gamma^0(\vec{\gamma}\cdot\vec{p} + mc)$ with $\vec{p} \equiv -i\hbar\vec{\partial}$

how do we get this?

SE: $i\hbar\partial_t\Psi = H\Psi$ and Dirac: $[i\hbar\gamma^\mu\partial_\mu - mc]\Psi = 0$

$\rightarrow i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = i\hbar\gamma^0\partial_0\Psi + i\hbar\gamma^j\partial_j\Psi - mc\Psi = 0$

$\Rightarrow i\hbar\gamma^0\frac{\partial}{c\partial t}\Psi = mc\Psi - i\hbar\gamma^j\partial_j\Psi$

$\vec{p} = i\hbar\vec{\partial}$
$\gamma_0\gamma_0 = \mathbb{1}$

$\Rightarrow H\Psi = c\gamma_0(mc + \gamma_j p_j)\Psi$

now for $\vec{L} = \vec{r}\times\vec{p}$: $[H, \vec{L}] = -i\hbar c\gamma^0(\vec{\gamma}\times\vec{p})$ thus $\vec{L}$ is not conserved

prove this

$[H, L_x] = [c\gamma^0(\vec{\gamma}\cdot\vec{p} + mc), yp_z - zp_y]$

$\vec{L}_i = (\vec{r}\times\vec{p})_i$

doesn't commute with constant $\gamma^0 mc^2$

$= c[\gamma^0(\vec{\gamma}\cdot\vec{p}), yp_z - zp_y]$

$= c\gamma^0\vec{\gamma}\cdot([\vec{p}, yp_z] - [\vec{p}, zp_y])$

$= c\gamma^0\vec{\gamma}\cdot([\vec{p}, y]p_z - [\vec{p}, z]p_y)$

$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$
$[m, n] = -[n, m]$

$= -c\gamma^0\vec{\gamma}\cdot(i\hbar p_z\hat{y} - i\hbar p_y\hat{z})$

$= -c\gamma^0(i\hbar p_z\gamma^2 - i\hbar p_y\gamma^3) = -i\hbar c\gamma^0(\vec{\gamma}\times\vec{p})_x$

spin ~ how something transforms under rotation
↳ Noether: invar under transfo
⟷ conserved quantity

- Dirac spinors

$$H = c\gamma^0(\vec{\gamma}\cdot\vec{p} + mc)$$

$$[H, \vec{L}] = -i\hbar c\gamma^0(\vec{\gamma}\times\vec{p})$$

⇒ Dirac field no spin 0 because all spin 0 fields have a conserved $\vec{L}$

↳ compensate non-conservation by introducing internal spin $\vec{S}$

for $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$ $\quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$\Rightarrow [H, \vec{L} + \vec{S}] = 0 \quad \rightarrow \vec{L} + \vec{S} = \vec{J}$ conserved

now $\vec{S}$ corresponds to spin 1/2

since $\vec{S}\times\vec{S} = i\hbar\vec{S}$

$\vec{S}\cdot\vec{S} = 3\hbar^2/4 \; \mathbb{1}$

Thus Dirac spinors = spin 1/2 particles

<u>derive this</u>

* $[p_i, S_j] = 0$ for $i, j \in \{1, 2, 3\}$
* $\vec{\Sigma}\gamma^0 = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} = \gamma^0\vec{\Sigma}$ for γ^0 Griffiths

$\Rightarrow [\vec{S}, \gamma^0] = 0$

* $i = 1$: $\Sigma_1\gamma^1 = \gamma^1\Sigma_1$

$\Sigma_1\gamma^2 - \gamma^2\Sigma_1 = 2i\gamma^3$

$\Sigma_1\gamma^3 - \gamma^3\Sigma_1 = -2i\gamma^2$

gebruik makend van commutatieregels Pauli

$\rightarrow [H, S_1] = \frac{\hbar c}{2}\gamma^0[\vec{\gamma}\cdot\vec{p}, \Sigma_1]$

$= \frac{\hbar c}{2}\gamma^0([\gamma^2, \Sigma_1]p_2 + [\gamma^3, \Sigma_1]p_3)$

$= ic\hbar\gamma^0(-\gamma^3 p_2 + \gamma^2 p_3)$

$= ic\hbar\gamma^0(\vec{\gamma}\times\vec{p})_1$

$\Rightarrow [H, \vec{L} + \vec{S}] = 0$

* check that the Griffiths basis works

...

![LMS Engineering Innovation]

3 RQM continued

transforming a Dirac spinor

- transformation matrix Dirac spinor ψ: S

  $\psi' = S\psi$

  demand Dirac eq. Lorentz invariant : Lorentz transfo : sol $\mapsto$ solution

  $\rightarrow \quad i\hbar (SS^{-1})\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S\psi - mcS\psi = 0 \qquad \partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$

  $\Rightarrow \quad S[i\hbar S^{-1}\gamma^\mu S \Lambda_\mu{}^\nu \partial_\nu - mc]\psi = 0$

  for it to be Lorentz invariant

  $S^{-1}\gamma^\mu S \Lambda_\mu{}^\nu \partial_\nu = \gamma^\mu \partial_\mu$

  $\Rightarrow \quad S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$

  $\eta \Lambda = \eta$

  $\Lambda^{-1} = \eta \Lambda^T \eta$

  solution in Weyl basis

  $S = \begin{pmatrix} A^{-+} & 0 \\ 0 & A \end{pmatrix}$ $\quad A$ = invertible complex matrix

  $A^{-+} = (A^+)^{-1} = (A^{-1})^+ \qquad (+ = T + *)$

  solution in Griffiths basis: $S_W = K S_G K^{-1}$

- Lorentz invariant combinations of spinors

  adjoint Dirac spinor : $\bar\psi = \psi^+ \gamma^0$

  using $S^+\gamma^0 S = \gamma^0$

  $\Rightarrow \bar\psi' = \bar\psi S^{-1}$ (transformation adjoint spinor under Lorentz rotation)

  check $S^+\gamma^0 S = \gamma^0$ in Weyl basis

  $S^+\gamma^0 = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^+ \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow S^+\gamma^0 S = \begin{pmatrix} 0 & A^{-1} \\ A^+ & 0 \end{pmatrix}\begin{pmatrix} A^{-+} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^0$

  $\bar\psi' = \bar\psi S^{-1}$

  $\bar\psi' = (S\psi)^+\gamma^0 = \psi^+ S^+ \gamma^0 = \psi^+ \gamma^0 S^{-1} = \bar\psi S^{-1}$

  then $\bar\psi\psi$ is a scalar and $\bar\psi\gamma^\mu\psi$ a vector

  show that $\bar\psi\gamma^\mu\psi$ transforms as a vector

  $(\bar\psi\gamma^\mu\psi)' = \bar\psi'\gamma^\mu\psi' = \bar\psi S^{-1}\gamma^\mu S\psi$

  $= \bar\psi \Lambda^\mu{}_\nu \gamma^\nu \psi = \Lambda^\mu{}_\nu \bar\psi\gamma^\nu\psi \rightarrow$ vect transfo

  QED Lagrangian that couples charged part. to Maxwell field

  $\mathcal{L}_I = \bar\psi \gamma^\mu \psi A_\mu$ $\quad$ Lorentz invariant quantity

  anti-symm tensor

  $\bar\psi \sigma^{\mu\nu} \psi \qquad$ with $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

  prove this

  tensor: $(\bar\psi\sigma^{\mu\nu}\psi)' = \bar\psi S^{-1}\sigma^{\mu\nu} S\psi = \frac{i}{2}\bar\psi(S^{-1}\gamma^\mu\gamma^\nu S - S^{-1}\gamma^\nu\gamma^\mu S)\psi$

  $= 1/2\, \bar\psi(S^{-1}\gamma^\mu S S^{-1}\gamma^\nu S - S^{-1}\gamma^\nu S S^{-1}\gamma^\mu S)\psi$

  $= 1/2\, \psi(\Lambda^\mu{}_\rho \gamma^\rho \Lambda^\nu{}_\sigma \gamma^\sigma - \Lambda^\nu{}_\sigma \gamma^\sigma \Lambda^\mu{}_\rho \gamma^\rho)\psi$

  $= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (\bar\psi \sigma^{\rho\sigma}\psi)$

  antisymm: $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$

define $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\quad$ where $\{\gamma^5, \gamma^\mu\} = 0$

in Griffiths basis

$\gamma^5_G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

in Weyl basis

$\gamma^5_W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

show this γ^5_W

$\gamma^i_W = \gamma^i_G \rightarrow \gamma^5_W = i\gamma^0_W\gamma^1_W\gamma^2_W\gamma^3_W$

$= i\gamma^0_W\gamma^1_G\gamma^2_G\gamma^3_G$

$= i\gamma^0_W\gamma^0_G\gamma^5_G$

$\rightarrow$ simple calculation

show this $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $\quad$ for $\mu \neq \nu$

$\rightarrow \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$

if $\mu = 0$: $\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu$

$= -i\gamma^0\gamma^\mu\gamma^1\gamma^2\gamma^3$

$= -\gamma^\mu\gamma^5$

$\Rightarrow \{\gamma^5, \gamma^\mu\} = 0$

holds for any μ since we will always have to switch a γ^μ with a γ^ν where $\mu \neq \nu$ 3 times : $(-1)^3 = -1$

transforming Dirac spinors

- pseudo scalar : $\bar{\Psi}\gamma^5\Psi$
  pseudo vector : $\bar{\Psi}\gamma^5\gamma^\mu\Psi$ — meaning that it gets a minus sign under parity transfo $\vec{x} \to -\vec{x}$

parity transfos are represented by $S=\gamma^0$

show this parity transfo : $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ → check that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$ holds for our $\Lambda^\mu{}_\nu$ and $S=\gamma^0$

* $\mu \neq \nu : \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$
  for $\mu \neq 0 : (\gamma^0)^{-1}\gamma^\mu\gamma^0 = -(\gamma^0)^{-1}\gamma^0\gamma^\mu = -\gamma^\mu$
* for $\mu = 0 : (\gamma^0)^{-1}\gamma^0\gamma^0 = \gamma^0 \; (=\gamma^\mu)$

} $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$ ok!

and that $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar

$(\bar{\Psi}\gamma^5\Psi)' = \bar{\Psi}S^{-1}\gamma^5 S\Psi = \bar{\Psi}(\gamma^0)^{-1}\gamma^5\gamma^0\Psi = i\bar{\Psi}(\gamma^0)^{-1}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\Psi$
$= i\bar{\Psi}\gamma^1\gamma^2\gamma^3\gamma^0\Psi = -\bar{\Psi}\gamma^5\Psi$

- spinor-bilinear : $\bar{\Psi}_1 M \Psi_2$; with M a matrix that acts on the spinor indices and Ψ_1, Ψ_2 spinor matrices

now $\bar{M} = \gamma^0 M^\dagger \gamma^0$ $\Rightarrow (\bar{\Psi}_1 M \Psi_2)^* = \bar{\Psi}_2 \bar{M} \Psi_1$

show this $\bar{\Psi}_2\bar{M}\Psi_1 = \Psi_2^\dagger\gamma^0\gamma^0 M^\dagger\gamma^0\Psi_1 = \Psi_2^\dagger M^\dagger\gamma^0\Psi_1 = (\bar{\Psi}_1 M\Psi_2)^\dagger = (\bar{\Psi}_1 M \Psi_2)^*$

show that $\bar{\gamma}^\mu = \gamma$

$\bar{\gamma}^\mu = \gamma^0(\gamma^\mu)^\dagger\gamma^0 = (\gamma^0\gamma^\mu\gamma^0)^\dagger = ((\gamma^\mu)^\dagger)^\dagger = \gamma^\mu$
or * $\mu \neq 0 : (\gamma^\mu)^\dagger = -\gamma^\mu$ and $\gamma^0\gamma^\mu\gamma^0 = -\gamma^\mu \to \bar{\gamma}^\mu = \gamma^\mu$
* $\mu = 0 : \bar{\gamma}^\mu = \gamma^0(\gamma^0)^\dagger\gamma^0 = \gamma^\mu$

and that the QED interaction $\mathcal{L}_I = \bar{\Psi}\gamma^\mu\Psi A_\mu$ is real

$\mathcal{L}^* = (\bar{\Psi}\gamma^\mu\Psi A_\mu)^* = (\bar{\Psi}\gamma^\mu\Psi)^* A_\mu^* = (\bar{\Psi}\gamma^\mu\Psi)^\dagger A_\mu^*$
$= (\bar{\Psi}\bar{\gamma}^\mu\Psi)^\dagger A_\mu = \Psi^\dagger(\bar{\gamma}^\mu)^\dagger\bar{\Psi}^\dagger A_\mu = \Psi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\Psi A_\mu$
$= \bar{\Psi}\gamma^\mu\Psi A_\mu = \bar{\Psi}\bar{\gamma}^\mu\Psi A_\mu$

probability currents

$j^\mu = \bar{\Psi}\gamma^\mu\Psi$ obeys $\partial_\mu j^\mu = 0$

show this Dirac: $i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0$ and $i\hbar\partial_\mu\bar{\Psi}\gamma^\mu + \bar{\Psi}mc = 0$

$\to i\hbar\partial_\mu j^\mu = i\hbar\bar{\Psi}\gamma^\mu\partial_\mu\Psi + (i\hbar\partial_\mu\bar{\Psi}\gamma^\mu)\Psi$
$= \bar{\Psi}mc\Psi - \bar{\Psi}mc\Psi = 0$

now if we interpret j^0 as prob. density ρ, we have that probability is conserved and ρ is manifestly positive : $j^0 = c\rho = c\bar{\Psi}\gamma^0\Psi = c\Psi^\dagger\Psi$, $j^\mu \to \vec{j}$ current

Dirac action and gauging principle

- Lagrangian density free Dirac field

$\mathcal{L} = i\hbar\bar{\Psi}\gamma^\mu\partial_\mu\Psi - mc\bar{\Psi}\Psi$ — **units** I don't know

$\to \frac{\partial\mathcal{L}}{\partial\bar{\Psi}} = i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0$ (Dirac eq.)

and $\partial_\mu \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Psi]} = i\hbar\partial_\mu\bar{\Psi}\gamma^\mu$

$\Rightarrow$ EL w.r.t. Ψ : $i\hbar\partial_\mu\bar{\Psi}\gamma^\mu + mc\bar{\Psi} = 0$ — Dirac conjugate Dirac eq.

Is this Lagrangian real? [Dirac]

$\mathcal{L}^* = -i\hbar(\partial_\mu\bar{\Psi})\gamma^\mu\Psi - mc\bar{\Psi}\Psi \neq \mathcal{L}$ → not real

Does this not contradict the statement that actions have to be real?

$\mathcal{L}^* - \mathcal{L}$ gives us the imaginary part

when looking at the action $S = \int d^4x\, \mathcal{L}^* = \int d^4x\, [(-i\hbar\partial_\mu\bar{\Psi})\gamma^\mu\Psi - mc\bar{\Psi}\Psi]$

we can solve the first part with partial integration → $\int d^4x (-i\hbar\partial_\mu\bar{\Psi})\gamma^\mu\Psi = -i\hbar\int d^4x\, \partial_\mu(\bar{\Psi}\gamma^\mu\Psi)$
$+ i\hbar\int d^4x\, \bar{\Psi}\gamma^\mu\partial_\mu\Psi$

$\int(\partial_\mu B^\mu)A^\mu = \underbrace{\int \partial_\mu(B^\mu A^\mu)}_{\text{total deriv}} - (\partial_\mu A^\mu)B^\mu$

thus $S = \int d^4x\, \mathcal{L}^* = \int d^4x\, \mathcal{L} + \int$ total deriv
$= \int d^4x\, \mathcal{L}$ → ok!

Lagrangian has to be real but not manifestly → imaginary part Lagrangian may be a total derivative → no difference

$\psi \to e^{i\alpha(t,x)}\psi$ then Dirac no longer invariant

- gauging principle

Dirac eq has global $U(1)$ invariance $\psi \to e^{iq\theta}\psi$ $0 \le \theta < 2\pi$, q some number

conservation charge $\partial_\mu j^\mu = 0$

gauging principle: a global symm can be made local by introducing a gauge field B^μ → does not depend on spacetime coord

local symm: θ arbitr. function "

$\theta(x^\mu)$: under local transfo:

$$\partial_\mu \psi \to e^{iq\theta}(\partial_\mu + iq\partial_\mu\theta)\psi$$

declare gauge field B^μ as new vector d.o.f that transforms the same

$$B_\mu \to B_\mu - \partial_\mu\theta$$

~ Lorentz gauge w $\theta = -\lambda$ "phase change"

⇒ write down gauge invariant Lagrangian by replacing

$\partial_\mu\psi \to D_\mu\psi$ where $D_\mu = \partial_\mu + iqB_\mu$ the covariant deriv.

⇒ adds $qB_\mu\bar{\psi}\gamma^\mu\psi$ to action

Yang-Mills theory (theorie gebied)

$i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi$
$\to i\hbar\bar{\psi}e^{-i\theta(x^\mu)}\gamma^\mu\partial_\mu(e^{i\theta}\psi)$
$\to \hbar\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\psi(i\partial_\mu\theta)i\hbar$

say $B_\mu = A_\mu$ (Maxwell field)

$$\to \mathcal{L} = \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell}} + \underbrace{i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - mc\bar{\psi}\psi}_{\text{Dirac free field}} - \underbrace{\overset{\text{charge}}{q}\hbar A_\mu\bar{\psi}\gamma^\mu\psi}_{\text{QED interaction term}}$$

not invariant term no dependence on A_μ $\lambda \cdot 0 \dots$

↳ coupling between a Maxwell field and a charged spinor

how would you describe a charged KG-field?

KG: $\hbar^2\Box\psi = m^2c^2\psi$ $\to \mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - (mc/\hbar)^2\psi^*\psi$

now consider $D_\mu = \partial_\mu + iqA_\mu$ and sub $\partial_\mu\psi \to D_\mu\psi$

$\Rightarrow \mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - (mc/\hbar)^2\psi^*\psi + q^2A_\mu\psi^*A^\mu\psi$

→ total Lagrangian dens:

$$\mathcal{L} = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - iqA_\mu)\psi^*(\partial^\mu + iqA^\mu)\psi - (mc/\hbar)^2\psi^*\psi$$

consider a coupling of a Maxwell field A to some current 4-vect j^μ through the following term in the action $A^\mu j_\mu$. What is needed for this term to have the equation of motion gauge invariant?

plane wave Dirac solutions

[Griffiths basis]

- simplest solution: no dependence spatial coord → delocalized part @ rest

for $\psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$ (4 comp spinor; 2 comp spinor) → $\partial_t\psi_A = -i(mc^2/\hbar)\psi_A$ → $\psi_A = \psi_A(0)\exp(-imc^2/\hbar\, t)$

$\partial_t\psi_B = +i(mc^2/\hbar)\psi_B$ $\psi_B = \psi_B(0)\exp(-imc^2/\hbar\, t)$

<u>verify this</u> Dirac: $i\hbar\gamma^\mu\partial_\mu\psi = mc\psi \to \gamma^\mu\partial_\mu\psi = -\frac{imc}{\hbar}\psi$

since no spatial dependence, only $\mu = 0$

$\gamma^0\frac{\partial}{\partial(ct)}\psi = \frac{-imc}{\hbar}\psi \to \gamma^0\partial_t\psi = -i(mc^2/\hbar)\psi$

for Griffiths: $\gamma^0 = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}$ → $\psi_A \sim \mathbb{1}$, $\psi_B \sim -\mathbb{1}$ $\left\{ \begin{array}{l}\partial_t\psi_A = -i(mc^2/\hbar)\psi_A \\ \partial_t\psi_B = i(mc^2/\hbar)\psi_B\end{array}\right.$

energy ~ fase factor: $\exp(-iE/\hbar\, t)$

⇒ ψ_A has rest energy mc^2

ψ_B " " " $-mc^2$ → ψ_B describes antiparticles with rest energy $+mc^2$

for ψ we have 4 solutions: 2 particles (ψ_A) x 2 antiparticles (ψ_B)

↳ 2 possible spins each time

$\Rightarrow \psi^{(1)} = \exp(-\frac{imc^2}{\hbar}t)(1\ 0\ 0\ 0)^T$, $\psi^{(2)} = \exp(-\frac{imc^2}{\hbar}t)(0\ 1\ 0\ 0)^T$

$\psi^{(3)} = \exp(imc^2/\hbar\, t)(0\ 0\ 1\ 0)^T$, $\psi^{(4)}\exp(imc^2/\hbar\, t)(0\ 0\ 0\ 1)^T$

- general plane wave solutions to a linear diff eq.

Ansatz: $\psi(x) = a\exp(-ik\cdot x)u(k)$ with $k\cdot x = k^\mu x^\nu \eta_{\mu\nu}$

→ split 4-comp $u(k)$ in 2 2-comp spinors u_A, u_B

$$u(k) = \begin{pmatrix}u_A(k) \\ u_B(k)\end{pmatrix}$$

- general plane wave solutions

$$\Psi(x) = a\exp(-ik\cdot x)\,u(k) \qquad \text{with } u(k) = \begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix}$$

↳ plug in Dirac using $\partial_\mu \Psi = -ik_\mu \Psi$

<u>show this</u> $\partial_\mu \Psi = a\,\partial_\mu \exp(-ik_\nu x^\nu)\,u(k)$

$$= a\prod_{j\neq\mu}\left[\exp(-ik^j x_j)\right]\frac{\partial}{\partial x^\mu}\exp(-ik_\mu x^\mu)\,u(k)$$

$$= -ik_\mu\, a\exp(-ik_\nu x^\nu)u(k) = -ik_\mu \Psi$$

$-i\hbar\gamma^\mu(-ik_\mu\Psi)$
$= mc\Psi$
$\Rightarrow \hbar\gamma^\mu k_\mu \Psi = mc\Psi$

$$\Rightarrow (\hbar\gamma^\mu k_\mu - mc)\,u(k) = 0 \qquad \text{with } \gamma^\mu k_\mu = \begin{pmatrix} k^0 & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix}1\\2\end{matrix}\begin{pmatrix} (\hbar k^0 - mc)u_A - \hbar\vec{k}\cdot\vec{\sigma}\,u_B \\ \hbar\vec{k}\cdot\vec{\sigma}\,u_A - (\hbar k^0 + mc)\,u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left[\hbar\begin{pmatrix} k^0 & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -k^0 \end{pmatrix} - mc\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\to \ 1)\ u_A = \frac{1}{k^0 - \frac{mc}{\hbar}}(\vec{k}\cdot\vec{\sigma})\,u_B \qquad 2)\ u_B = \frac{1}{k^0 + \frac{mc}{\hbar}}(\vec{k}\cdot\vec{\sigma})\,u_A$$

2) in 1)

$$\to \quad u_A = \frac{(\vec{k}\cdot\vec{\sigma})^2 u_A}{(k^0)^2 - (mc/\hbar)^2}$$

$$\Rightarrow \frac{\vec{k}^2}{(k^0)^2 - (mc/\hbar)^2} = 1 \quad \Leftrightarrow \quad k^2 = \left(\frac{mc}{\hbar}\right)^2$$

<u>check this</u> $u_A = \frac{(\vec{k}\cdot\vec{\sigma})^2}{(k^0)^2 - (mc/\hbar)^2}\,u_A \Rightarrow 1 = \frac{(\vec{k}\cdot\vec{\sigma})^2}{(k^0)^2 - (mc/\hbar)^2}$

and $(\vec{k}\cdot\vec{\sigma})^2 = (k^1)^2\sigma^1\sigma^1 + (k^2)^2\sigma^2\sigma^2 + (k^3)^2\sigma^3\sigma^3$

and since $\sigma^i\cdot\sigma^i = 1$

we find $(\vec{k}\cdot\vec{\sigma})^2 = (k^1)^2 + (k^2)^2 + (k^3)^2 = \vec{k}^2$

↳ using $\partial_\mu \Psi = -ik_\mu \Psi$: $\hbar k^\mu = p^\mu$ [since $\partial_\mu \Psi = -i\hbar p_\mu \Psi$]

or $\hbar k^\mu = -p^\mu$ (also allowed)

↳ 4 solutions:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_B = \frac{1}{k^0 + \frac{mc}{\hbar}}(\vec{k}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_B = \frac{1}{k^0 + mc/\hbar}(\vec{k}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_A = \frac{1}{k^0 - mc/\hbar}(\vec{k}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_A = \frac{1}{k^0 - mc/\hbar}(\vec{k}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Psi_A = a\exp(-ik\cdot x)\,u_A(k)$
$\to p^\mu = \hbar k^\mu$

$\Psi_B = a\exp(-ik\cdot x)\,u_B(k)$
$\to p^\mu = -\hbar k^\mu$ (to fix neg. energy)

so that for zero momentum we get solutions from earlier

- ambiguity Dirac

comes from the sign ambiguity $i\hbar\gamma^\mu\partial_\mu\Psi \mp mc\Psi = 0$

choosing $i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0$ forces us to chose $p_\mu = -i\hbar\partial_\mu$ for antiparticles

(we chose the ⊖ but when deriving dirac the ⊕ was also an option)

- normalization spinors

$$\bar{\Psi}\gamma^0\Psi = \Psi^\dagger\Psi = 2E/c$$

$u^{(2)} = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$\Rightarrow u^{(1)} = N\left(1 \quad 0 \quad \frac{cp_z}{E+mc^2} \quad \frac{c(p_x+ip_y)}{E+mc^2}\right)^T, \quad u^{(2)} = N\left(0 \quad 1 \quad \frac{c(p_x-ip_y)}{E+mc^2} \quad \frac{-cp_z}{E+mc^2}\right)^T$$

$$v^{(1)} = N\left(\frac{c(p_x-ip_y)}{E+mc^2} \quad \frac{-p_z c}{E+mc^2} \quad 0 \quad 1\right)^T, \quad v^{(2)} = -N\left(\frac{cp_z}{E+mc^2} \quad \frac{c(p_x+ip_y)}{E+mc^2} \quad 1 \quad 0\right)^T$$

↳ convention

with $N = \sqrt{\frac{E+mc^2}{c}}$

<u>verify</u> how we got to the expressions of the explicit spinors and the normalization

$\sigma^1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad\sigma^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$ $\quad\sigma^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\sigma^1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad\sigma^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$ $\quad\sigma^3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\to u^{(1)}: \ u_B = \frac{1}{k^0 + mc/\hbar}\sum_{i=1}^3 k^i\sigma^i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{p^0 + mc}\sum_{i=1}^3 p^i\sigma^i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{p^0 + mc}\left[\begin{pmatrix} 0 \\ p_x \end{pmatrix} + \begin{pmatrix} 0 \\ ip_y \end{pmatrix} + \begin{pmatrix} p_z \\ 0 \end{pmatrix}\right] = \frac{c}{E + mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

analogous for $u^{(2)}, v^{(1)}, v^{(2)}$

- normalization spinors

verify for $u^{(1)}$: $(u^{(1)})^+ u^{(1)} = 2E/c$

$$\rightarrow (u^{(1)})^+ u^{(1)} = N^2 \left(1 \quad 0 \quad \frac{cp_z}{E+mc^2} \quad \frac{c(p_x - ip_y)}{E+mc^2}\right) \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

$$= N^2 \left[1 + \frac{c^2}{(E+mc^2)^2} (p_z^2 + p_x^2 + p_y^2)\right]$$

$|\vec{p}|^2 = (E/c)^2 - m^2c^2$

$$= N^2 \left(1 + \frac{E^2 - m^2c^4}{(E+mc^2)^2}\right)$$

$$= N^2 \left(1 + \frac{(E-mc^2)(E+mc^2)}{(E+mc^2)^2}\right)$$

$$= N^2 \left(1 + \frac{E-mc^2}{E+mc^2}\right) = N^2 \frac{2E}{E+mc^2}$$

$$= \frac{2E}{c} \Rightarrow N = \pm\sqrt{\frac{E+mc^2}{c}}$$

$\Rightarrow$ particles: $\Psi = a \exp(-ip\cdot x/\hbar)\, u \qquad \rightarrow (\gamma^\mu p_\mu - mc\mathbb{1}) u = 0$

anti-" : $\Psi = a \exp(ip\cdot x/\hbar)\, v \qquad \rightarrow (\gamma^\mu p_\mu + mc\mathbb{1}) v = 0$ **

with orthogon. en norm relations $\quad \hookrightarrow \bar{u}(\gamma^\mu p_\mu - mc) = 0$ *

taking + and then $\cdot\gamma^0$ $\quad \bar{v}(\gamma^\mu p_\mu + mc) = 0$

$(\gamma^\mu p_\mu)^+ = \ldots$

1. $\bar{u}^{(1)} u^{(2)} = \bar{v}^{(1)} v^{(2)} = 0$
2. $\bar{u}^{(1)} u^{(1)} = \bar{u}^{(2)} u^{(2)} = 2mc$
3. $\bar{u} v = \bar{v} u = 0$
4. $\bar{v}^{(1)} v^{(1)} = \bar{v}^{(2)} v^{(2)} = -2mc$

verify

3) $\bar{u}^{(r)}(\vec{p})\, \not{p}\, v^{(s)}(\vec{p}) \qquad$ with $\not{p} = \gamma_\mu p^\mu$

$(\bar{u}^{(r)}(\vec{p})\not{p})\, v^{(s)}(\vec{p}) \overset{*}{=} \bar{u}^{(r)}(\vec{p})\, mc\, v^{(s)}(\vec{p}) = mc\, \bar{u}^{(r)}(\vec{p}) v^{(s)}(\vec{p})$] ?

$\bar{u}^{(r)}(\vec{p})(\not{p}\, v^{(s)}(\vec{p})) \overset{**}{=} \bar{u}^{(r)}(\vec{p})(-mc)\, v^{(s)}(\vec{p}) = -mc\, \bar{u}^{(r)}(\vec{p}) v^{(s)}(\vec{p})$

$\Rightarrow \bar{u}^{(r)}(\vec{p}) v^{(s)}(\vec{p}) = 0$

analogous for $\bar{v} u = 0$

1) $\bar{u}^{(r)}(\vec{p})\, \sigma_{\vec{p}}\, u^{(s)}(\vec{p}) \qquad$ with $\sigma_{\vec{p}} = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{\sigma}\cdot\vec{p}}{p_0}$

?

$(\bar{u}^{(r)} \frac{\vec{\sigma}\cdot\vec{p}}{p_0}) u^{(s)} = \ldots$

$\bar{u}^{(r)} (\frac{\vec{\sigma}\cdot\vec{p}}{p_0} u^{(s)}) = \ldots$

$\Rightarrow \bar{u}^{(r)} u^{(s)} = 0$

PAL

analogous for $\bar{v} v$

4) $\bar{v}^{(r)}(\vec{p})(\not{p} + mc) v^{(m)}(\vec{p}) = 0$

$\Rightarrow \ldots$

geen idee

completeness relation

$$\sum_{s=1,2} u_\alpha^{(s)} \bar{u}_\beta^{(s)} = (\gamma^\mu p_\mu + mc\mathbb{1})_{\alpha\beta}$$

$$\sum_{s=1,2} v_\alpha^{(s)} \bar{v}_\beta^{(s)} = (\gamma^\mu p_\mu - mc\mathbb{1})_{\alpha\beta}$$

prove this — see back of this page

[we've found 4 part. states but expected 2 since spin 1/2
other 2: antipart

- normalization spinors

prove completeness relations

we choose to work in the rest frame $p_x = p_y = p_z = 0$ (we can choose this since the particles have mass)

$$\sum_{s=1,2} u_\alpha^{(s)}(\vec{p})\,\bar{u}_\beta^{(s)}(\vec{p}) = (\gamma^\mu p_\mu + mc\mathbb{1})_{\alpha\beta}$$

LHS $u^{(1)}(\vec{p}) = N\,(1\;0\;0\;0)^T$ $\quad \bar{u}u = 2mc$ $\quad u^{(2)}(\vec{p}) = N(0\;1\;0\;0)^T$

$\bar{u}^{(1)}(\vec{p}) = \frac{2mc}{N}(1\;0\;0\;0)$ $\quad \bar{u}^{(2)}(\vec{p}) = \frac{2mc}{N}(0\;1\;0\;0)$

$$\Rightarrow u^{(1)}(\vec{p})\bar{u}^{(1)}(\vec{p}) + u^{(2)}(\vec{p})\bar{u}^{(2)}(\vec{p})$$

$$= 2mc\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + 2mc\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} = 2mc\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

RHS $(\gamma^\mu p_\mu + mc\mathbb{1})_{\alpha\beta}$

$$= \gamma^\mu p_\mu + mc\,\mathbb{1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}mc + mc\,\mathbb{1}$$

$\begin{cases} p_1 = p_2 = p_3 = 0 \\ p_0 = mc \\ \gamma^0 = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\end{cases}$ $\rightarrow$

$$= mc\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} = 2mc\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$\rightarrow$ RHS = LHS in rest frame

now since all elements act the same under Lorentz transfos

it holds in all frames

↳ analogous for v

alternative

$u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ form a basis

thus if we let $\sum_{s=1,2} u_\alpha^{(s)}(\vec{p})\,\bar{u}_\beta^{(s)}(\vec{p}) = (\gamma^\mu p_\mu + mc\mathbb{1})_{\alpha\beta}$ on this basis, we get

$u^{(1)}$: $\sum_{s=1,2} u_\alpha^{(s)}\,\bar{u}_\beta^{(s)}\,u^{(1)} \stackrel{!}{=} (\gamma^\mu p_\mu + mc\mathbb{1})\,u^{(1)}$

$\rightarrow u_\alpha^{(1)}\bar{u}_\beta^{(1)}u^{(1)} = (\gamma^\mu p_\mu + mc\,\mathbb{1})\,u^{(1)}$

RHS : $\not{p}\,u^{(1)} + mc\,u^{(1)} = mc\,u^{(1)} + mc\,u^{(1)}$

$= 2mc\,u^{(1)}$

LHS : $u_\alpha^{(1)}\bar{u}_\beta^{(1)}u^{(1)} = u_\alpha^{(1)}\,2mc = 2mc\,u^{(1)}$ $\Big] =$

$u^{(2)}$: $u_\alpha^{(2)}u_\beta^{(2)}u^{(2)} = (\gamma^\mu p_\mu + mc\mathbb{1})\,u^{(2)}$

RHS $\not{p}\,u^{(2)} + mc\,u^{(2)} = 2mc\,u^{(2)}$ $\Big] =$

LHS $2mc\,u^{(2)}$

$v^{(1)}$: $\sum_{s=1,2} u_\alpha^{(s)}\,\bar{u}_\beta^{(s)}\,v^{(1)} = (\gamma^\mu p_\mu + mc\mathbb{1})\,v^{(1)}$

RHS $\not{p}\,v^{(1)} + mc\,v^{(1)} = -mc\,v^{(1)} + mc\,v^{(1)} = 0$ $\Big] =$

LHS 0 $\quad (\bar{u}v = 0)$

$v^{(2)}$: $\sum_{s=1,2} u_\alpha^{(s)}\,\bar{u}_\beta^{(s)}\,v^{(2)} = (\gamma^\mu p_\mu + mc\mathbb{1})\,v^{(2)}$

RHS $= 0$ $\Big] =$

LHS $= 0$

$\Rightarrow$ completeness relation holds

↳ analogous for v

4 Decay rates

decay rates

$\Gamma_i = c/s$, possibly different rates for different decay channels

→ for N unstable particles

average number decreases after dt : $\delta N = -N\, \delta t\, \Sigma_i \Gamma_i$ (decay)

with $\Gamma = \Sigma_i \Gamma_i$ the total decay rate $[1/s]$

→ amount of unstable part : $N(t) = N(0) \exp(-\Gamma t)$

⇒ average lifetime $\tau = 1/\Gamma$

<u>show this</u> particle decay ~ poisson distributed

thus # of decays in a certain time ~ Poisson

⇒ time between two Poisson-events ~ exponentially distr.

for $e(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ ⇒ $\mu = 1/\lambda$ (theory exp. distr)

thus for $N(t) = N(0)\exp(-\Gamma t)$: $\tau = 1/\Gamma$.

branching ratio = how much a certain decay channel (j) contributes to the total decay

: $BR(j) = \Gamma_j / \Gamma$

golden rule

- QM : Fermi's golden rule → decay rate

ex. this's a trick for decay since it is Poisson distributed and thus "has no memory" → prob for part decay independent of history

particle decays if it's not in its eigenstate H

→ $t < 0$: $H = H_0$ with $|i\rangle$ = initial state an eigenstate

$t \geq 0$: $H = H_0 + H'$ perturbation; <u>interaction H'</u> small compared to free H_0

want to know the transition rate Γ_{fi} from $|i\rangle$ to $|f\rangle$ (final state)

Fermi: $\Gamma_{fi} = \frac{2\pi}{\hbar} |T_{fi}|^2 \rho(E_i)$

perturb. not exact

→ 0 , anders interacties sneller dan licht

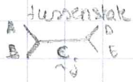
eerst overgang andere state die dan overlapt met de initiële state

approximation to n^st order only for infty

with $T_{fi} = \langle f|H'|i\rangle + \sum_{j \neq i} \frac{\langle f|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j} + \ldots$ the transition matrix element

→ j can be not unique → sum integration

tussenstate



and $\rho(E_i)$ density of states at initial energy E_i

<u>check dimensions</u>

$[T_{fi}]^2 = [E^2]$, $[\rho(E_i)] = [E^{-1}]$, $[\hbar] = [E] \cdot s$

⇒ $[\Gamma_{fi}] = [E]^2 [E]^{-1} [E]^{-1} s^{-1} = 1/s$ OK!

density of states

- $\rho(E) = \left|\frac{dn}{dE}\right|_E$ with dn = # accessible states with energy between E and dE

→ $\rho(E_i) = |dn/dE|_{E_i} = \int \frac{dn}{dE} \delta(E_i - E) dE$

⇒ Fermi: $\Gamma_{fi} = \frac{2\pi}{\hbar} \int |T_{fi}|^2 \delta(E_i - E)\, dn$

→ integrate over poss. mom × possible spins with $|T_{fi}|^2$ possibly depending on spin or mom

[if spin coupling: ·m]

- in derivation normalization wave-functions used

$\langle f|f\rangle = \langle i|i\rangle = 1$

→ for spin 0 in vol. V $\int_V dx^3 \Psi^*(x_1, x_2, x_3)\Psi(x_1, x_2, x_3) = 1$

consider $V = a^3$ (cubic box)

and free Hamiltonian $H_0 = p^2/2m$

eigenlijk niet norm. voor vrije deeltjes, doen alsof universum volume V, ∞ heeft

plane-wave mom eigstates as basis quantum states: $\Psi = N \exp\left(\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right)$

→ $N \sim 1/\sqrt{V}$

norm → $|N|^2 = a^{-3}$

now using periodic boundary conditions $\Psi(x+a, y, z) = \Psi(x,y,z)$

we get quantization $\vec{p} = \frac{2\pi\hbar}{a}\vec{n}$ $\vec{n} = (n_1, n_2, n_3)$, $n_i \in \mathbb{Z}$

- derivation density of states
  $\Psi = a^{-3/2} \exp\left(\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right) \xrightarrow{B.C} \vec{p} = \frac{2\pi\hbar}{a}\vec{x}$
  now $a \to \infty$ since $\rho(E)$ is a continuous function
  $\Rightarrow \Delta p_x \Delta p_y \Delta p_z = \left(\frac{2\pi\hbar}{a}\right)^3 = \frac{(2\pi\hbar)^3}{V}$

density of states
  we have $E = |\vec{p}|^2/2m$ → count # states Δn in a given interval for the momenta divided by the volume of a single quantumstate $\hookrightarrow \frac{(2\pi\hbar)^3}{V}$

momenta gekwantiseerd → kwantumtoest. vormen een rooster
kwantumtoest. 1/e volume $\frac{\text{Vol}}{\left(\frac{(2\pi\hbar)^3}{V}\right)}$

$\Delta n = \Delta p_x \Delta p_y \Delta p_z \frac{V}{(2\pi\hbar)^3}$
→ volume
$\to dn = dp_x dp_y dp_z \; V/(2\pi\hbar)^3$

now $dn(|p|) = 4\pi |p|^2 dp \; V/(2\pi\hbar)^3$ (spherical coord) suppose rot. symm (free part)
$\hookrightarrow$ number of states in interval $[|p|, |p|+d|p|]$

$\Rightarrow \rho(E) = \left|\frac{dn}{dE}\right| = \frac{dn}{d|p|}\left|\frac{d|p|}{dE}\right| = 4\pi |p|^2 \frac{V}{(2\pi\hbar)^3}\left|\frac{d|p|}{dE}\right|$

$\hookrightarrow$ higher momentum → more quantum states
  ⇒ favorable to decay to lightest states allowed
  since there are more quantum states for them → higher probability

- decay proces $A \to B + C$
  thus $|i\rangle = \Psi_A$, $|f\rangle = \Psi_B \Psi_C$ — normalized according to $\langle f|f\rangle = \langle i|i\rangle = 1$
  $\Rightarrow T_{fi} = \int_V \underbrace{\Psi_B^*}_{N^*} \underbrace{\Psi_C^*}_{N^*} H' \underbrace{\Psi_A}_{N} \sim \int_V \frac{1}{V} N \, dV \sim N$
  $\longrightarrow |N|^2 = a^{-3} = 1/V$

  $\to |T_{fi}|^2 \sim |N|^2 = 1/V$

  thus Fermi's golden rule for $A \to B + C$:
    Γ_{fi} V-independent since $|T_{fi}| \sim \frac{1}{V}$, $\rho \sim V$

decay to M particles: $A \to B_1 + \ldots + B_M$
  momentum decay products no longer uniquely fixed
  → dn_j = # quantum states for j^{th} decay prod.
  since conservation mom: M^{th} mom. can be written in terms of the other $M-1$ momenta
  → independent momentum quantum states = those of the $M-1$ particles
  $\to dn = \prod_{j=1}^{M-1} dn_j = \prod_{j=1}^{M-1} dp_x^{(j)} dp_y^{(j)} dp_z^{(j)} \frac{V}{(2\pi\hbar)^3}$
  $= \frac{(2\pi\hbar)^3}{V} \prod_{j=1}^{M} dp_x^{(j)} dp_y^{(j)} dp_z^{(j)} \frac{V}{(2\pi\hbar)^3} \underbrace{\delta\left(\vec{p}^{(A)} - \sum_{i=1}^{M} \vec{p}^{(i)}\right)}_{\text{conservation of momentum}}$

<u>explain</u> how the factors of V get cancelled in the equation expressing Fermi's golden rule

  $dn \sim V^{M-1}$
  $T_{fi} \sim \left(\frac{1}{\sqrt{V}}\right)^{M-1}$ since for 2 particles $|T_{fi}|^2 \sim V^{-1}$
  $\Rightarrow \Gamma_{fi} = \frac{2\pi}{\hbar}\int |T_{fi}|^2 \delta(E_i - E)\, dn$ → independent of V

relativistic treatment

- $H \to$ relativistic H (e.g. Dirac Hamiltonian)
- wave-function normalized: 1 particle per volume V : $\int_V |\Psi|^2 = 1$ ~ particle density
  → γ particles per volume V for boosted ref. frame (length contr.) → volume 1 part V/γ
  since densities always increased due to length contr.
  → thus for a Lorentz-invariant normalization
    $\langle \Psi'|\Psi'\rangle = \int_V \Psi'^* \Psi' = 2E/c$ ~ part. dens ($\gamma = E/mc^2$)
    $\hookrightarrow \frac{2E}{c}$ particles per volume V
    notice $\Psi' \neq \Psi$, $\Psi' = \sqrt{2E/c}\,\Psi$

  ⇒ transition matrix element
    $M_{fi} = \langle \Psi_1' \Psi_2' \ldots | H' | \Psi_a' \Psi_b' \ldots\rangle = \sqrt{\frac{2E_1}{c}\frac{2E_2}{c}\ldots\frac{2E_a}{c}\frac{2E_b}{c}\ldots}\; T_{fi}$
    which is Lorentz invariant

- normalized relativistic wave functions

for $a \to 1+2$

$$\Gamma_{ji} = \frac{2\pi}{\hbar}\int |T_{ji}|^2 \delta(E_a - E_1 - E_2)\, dn$$

$$= \frac{(2\pi)^4 \hbar^2 c^3}{2E_a}\int |M_{ji}|^2 \delta(E_a - E_1 - E_2)\,\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi\hbar)^3 2E_1}\,\frac{d^3\vec{p}_2}{(2\pi\hbar)^3 2E_2}$$

↳ whole integral should be Lorentz-invariant since prefactor $(E^a)^{-1}$ transforms as an inverse time-interval

→ integral over all 4 coords

using $\int \delta(p_i^2 - m_i^2c^2)\,\theta(p_i^0)\,dp_i^0 = \frac{c}{2E_i}$ w θ heaviside step-function

mass-shell condition

derive this

$$\int \delta(p_i^2 - m_i^2c^2)\,\theta(p_i^0)\,dp_i^0 = \int \delta((p_i^0)^2 - (\vec{p}_i)^2 - m_i^2c^2)\,\theta(p_i^0)\,dp_i^0$$

$$= \int \frac{\delta(p_i^0 - E_i/c) + \delta(p_i^0 + E_i/c)}{2E_i/c}\,\theta(p_i^0)\,dp_i^0 \quad ▲$$

$$= \frac{c}{2E_i} \qquad \text{only positive } p_i^0$$

▲ since $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i^*)|}\delta(x - x_i^*)$ with $g(x^*) = 0$

thus $\delta((p_i^0)^2 - (\vec{p}_i)^2 - m^2c^2) = \delta(g(p_i^0))$ → $p_i^{0*} = \pm\sqrt{\vec{p}_i^2 + m_i^2c^2} = E_i/c$

→ $g'(p_i^0) = 2p_i^0$

$$\Rightarrow \Gamma_{ji} = \frac{1}{2E_a(2\pi)^2\hbar^4}\int |M_{ji}|^2 \delta^4(p_a - p_1 - p_1)\,\delta(p_1^2 - m_1^2c^2)\,\theta(p_1^0)\,d^4p_1\,\delta(p_2^2 - m_2c)\,\theta(p_2^0)\,d^4p_2$$

verify

to be able to perform $c/2E_i$

$$\Gamma_{ji} = \frac{(2\pi)^4\hbar^2 c^3}{2E_a}\int |M_{ji}|^2 \underbrace{\delta(E_a - E_1 - E_2)\,\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)}_{*\ \delta^4(p_a - p_1 - p_2)\,1/c}\frac{d^3\vec{p}_1}{(2\pi\hbar)^3 2E_1}\,\frac{d^3\vec{p}_2}{(2\pi\hbar)^3 2E_2}$$

using $\frac{c}{2E_i} = \int \delta(p_i^2 - m_i^2c^2)\,\theta(p_i^0)\,dp_i^0$

$* \ \delta(\alpha x) = \frac{\delta(x)}{|\alpha|} \to \delta(E) = \delta(p^0c) = \delta p^0/c$

$$= \frac{(2\pi)^4\hbar^2}{2E_a}\,\frac{1}{(2\pi\hbar)^6}\int |M_{ji}|^2 \delta^4(p_a - p_1 - p_2)\,\delta(p_1^2 - m_1^2c^2)\,\theta(p_1^0)\,d^4p_1 \cdot \delta(p_2^2 - m_2^2c^2)\,\theta(p_2^0)\,d^4p_2 \quad \text{OK!}$$

↳ integral can be carried out explicitely since conserv. of mom. fixes the momenta of outgoing part and we have the δ-functions

$$\Rightarrow \Gamma = \frac{p^*}{8\pi m_a^2 \hbar^4 c^3}|M|^2 \quad \text{with } p^* = \frac{c}{2m_a}\sqrt{[m_a^2 - (m_1 + m_2)^2][m_a^2 - (m_1 - m_2)^2]}$$

↳ size outgoing mom. of either part 1 or 2 in CM frame

verify p^* → see back of page

↳ closer look integral

→ $\theta(x) = \begin{cases} 0 & x<0 \\ 1 & x \geq 1 \end{cases}$

$$\Gamma_{ji} = \underbrace{\frac{1}{2E_a(2\pi)^2\hbar^4}}_{\text{not Lorentz invariant}}\int \underbrace{|M_{ji}|^2}_{\substack{\text{Lorentz-invar}\\\text{dynamics}}}\underbrace{\underbrace{\delta^4(p_a^\mu - p_1^\mu - p_2^\mu)}_{\text{cons. mom.}}\underbrace{\delta((p_1^\mu)^2 - m_1^2c^2)\,\delta((p_2^\mu)^2 - m_2^2c^2)}_{\text{outgoing part. satisfy Einstein}}\underbrace{\theta(p_1^0)\,\theta(p_2^0)}_{\text{pos. energy (move forward in time)}}}_{\text{kinematics}}\,d^4p_1\,d^4p_2$$

↳ sum 4-vect → Lorentz invar

↳ timelike stays timelike Lorentz invar (forward in time)

not Lorentz invariant

$\frac{1}{E_a}$ transforms the same way as Γ_{ji} should verify p^*

performing integral

$$\int |M_{ji}|^2 \delta(E_a - E_1 - E_2)\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi\hbar)^3 2c\underbrace{\sqrt{\vec{p}_1^2+m_1^2c^2}}_{E_1}} \frac{d^3\vec{p}_2}{(2\pi\hbar)^3 2c\underbrace{\sqrt{\vec{p}_2^2+m_2^2c^2}}_{E_2}}$$

assume particle at rest

$\vec{p}_a = 0$, $p_a^0 = m_a c$ $\rightarrow \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) = \delta^3(\vec{p}_1 + \vec{p}_2) \Rightarrow \vec{p}_1 = -\vec{p}_2$

$\hookrightarrow$ integral $\displaystyle \frac{1}{4c^2(2\pi\hbar)^6}\int |M_{ji}|^2 \frac{\delta(m_ac^2 - c\sqrt{\vec{p}_1^2+m_1^2c^2} - c\sqrt{\vec{p}_1^2+m_2^2c^2})}{\sqrt{\vec{p}_1^2+m_1^2c^2}\sqrt{\vec{p}_1^2+m_2^2c^2}} d^3\vec{p}_1$

spherical coord where $r = |\vec{p}_1|$

$\hookrightarrow \displaystyle \frac{1}{4c^2(2\pi\hbar)^6}\int |M_{ji}|^2 \frac{\delta(m_ac^2 - c\sqrt{r^2+m_1^2c^2} - c\sqrt{r^2+m_2^2c^2})}{\sqrt{r^2+m_1^2c^2}\sqrt{r^2+m_2^2c^2}} dr\, d\Omega$

change of radial coord : $u = \sqrt{r^2+m_1^2c^2} + \sqrt{r^2+m_2^2c^2}$

$$\frac{du}{dr} = \frac{u\, r(u)}{\sqrt{r^2+m_1^2c^2}\sqrt{r^2+m_2^2c^2}}$$

$\hookrightarrow \displaystyle \frac{1}{4c^2(2\pi\hbar)^6}\int |M_{ji}|^2 \delta(m_ac^2 - uc)\frac{r(u)}{u} du\, d\Omega$

$$= \frac{1}{4c^3(2\pi\hbar)^6}\frac{r(u = m_ac^2)}{m_a c^2}\int |M_{ji}|^2 d\Omega$$

assuming M only depends on the size of the momentum

$\hookrightarrow \displaystyle \frac{\pi}{c^3(2\pi\hbar)^6}\frac{r(u=m_ac^2)}{m_ac^2}|M_{ji}|^2$

now $p^* = r(u = m_ac^2)$

thus : $u = \sqrt{r^2+m_1^2c^2} + \sqrt{r^2+m_2^2c^2} = \sqrt{r^2+a^2} + \sqrt{r^2+b^2}$ $\quad \begin{cases} a^2 = m_1^2c^2 \\ b^2 = m_2^2c^2 \end{cases}$

say $x = \sqrt{r^2+a^2} \rightarrow \sqrt{r^2+b^2} = \sqrt{x^2-a^2+b^2}$

$\Rightarrow u = x + \sqrt{x^2-a^2+b^2}$

$\Rightarrow u - x = \sqrt{x^2-a^2+b^2}$

$\Rightarrow u^2 - 2xu + \cancel{x^2} = \cancel{x^2} - a^2 + b^2$

$\Rightarrow x = \dfrac{u^2-b^2+a^2}{2u} = \sqrt{r^2+a^2}$

$\rightarrow r^2 = x^2 - a^2 = \dfrac{1}{4u^2}\left[u^4 + b^4 + a^4 - 2u^2b^2 - 2b^2a^2 + 2u^2a^2\right] - a^2$

$= \dfrac{1}{4u^2}\left[u^4 + b^4 + a^4 - 2u^2b^2 - 2b^2a^2 - 2u^2a^2\right]$

$= \dfrac{1}{4u^2}(u^2 - (a+b)^2)(u^2 - (a-b)^2)$

$\rightarrow p^* = \dfrac{1}{2m_ac}\sqrt{(m_a^2c^2 - (m_1c + m_2c)^2)(m_a^2c^2 - (m_1c - m_2c)^2)}$

take all c's out $\rightarrow$ eq. in book

$u = m_ac$

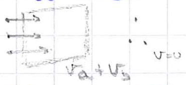
5 Cross section and amplitudes [scattering]

definition differential cross section

- scattering experiment

A○ $\xrightarrow{a}$ $\vec{v}_a$ ⠀⠀⠀ • b ⟷ $\vec{v}_B$ (collinear, but no direction specified)

beam w luminosity/flux ϕ_a and cross-sectional area A
flux = # part moving through a unit area in a unit time
particle densities n_a, n_b → total # particles N_a, N_b
→ $\phi_a = n_a(v_a + v_b)$ (flux)

explain  $\phi_a = n_a(v_a + v_b)$ → look at frame in which the target particles are at rest
density = $\frac{\text{part}}{\text{volume}}$ speed = $\frac{\text{length}}{\text{time}}$ → $\frac{\text{part}}{\text{area time}}$

→ interaction rate per target particle, r_b
$r_b = \sigma \phi_a$ with σ the cross-section
units $[\phi] = \frac{\text{part}}{m^2 \cdot s}$ $[\sigma] = m^2$ → $[r_b] = \text{part}/s$

total interaction rate r
$r \equiv r_b N_b \equiv r_a N_a$

cross section ~ effective size target particle
≡ size of the beam that is scattered by the target particle
however for interactions that don't die off fast enough
(e.g. EM: die off to ∞ without ever being 0) ⇒ $\sigma \to \infty$
⇒ differential cross section D
$D = d\sigma / d\Omega$

- interaction rate

in a time Δt a particle type a passes $\Delta N_b = \overset{= \phi_b A \Delta t}{(v_a + v_b) A n_b \Delta t}$ particles type b
σ = effective area around single part b ("interaction area")
→ part. a crossing σ will interact
⇒ interaction prob.: $\Delta P_a = \frac{\Delta N_b \sigma}{A} = n_b(v_a + v_b)\sigma \cdot \Delta t$
→ tot cross section area of target particles
→ tot cross section area of "a" particles

→ interaction rate for a single part a
$r_a = dP_a/dt = n_b(v_a + v_b)\sigma$
total interaction rate for all particles in tot. volume V
$r = r_a n_a V = n_b(v_a + v_b)\sigma n_a V = [n_a(v_a + v_b)][n_b V]\sigma = \phi_a N_b \sigma$
⇒ $r_b = r/N_b = \phi_a \sigma$

- differential cross section

when target is 'soft': interaction = process in which part a feels the presence of a force field that couples to part b
more than a collision
→ always non-zero interaction ⇒ $\sigma \to \infty$
⇒ differential cross section D = # part. scattered into solid angle $d\Omega$ per unit time per target part divided by ϕ_a

$$D\,d\Omega = \frac{\#\text{particles scattered into } d\Omega}{\Delta t N_b \phi_a}$$

$d\Omega = \sin\theta \, d\theta \, d\phi$ ⠀ $0<\phi<2\pi, 0<\theta<\pi$
→ integrate over area $d\Omega$, no $r^2 dr$ component

→ $\sigma = \int D \, d\Omega$ ⠀ or ⠀ $D = d\sigma/d\Omega$

relativistic treatment

- scattering event $a + b \to 1 + 2$

with $r = \phi_a N_b V \sigma$
normalize wave function as 1 part per volume
$\int |\psi|^2 = 1$ ⇒ $\Gamma_{fi} = r$, $n_a = n_b = 1/V$

- scattering event $a + b \to 1 + 2$

$\phi_a = (v_a+v_b) n_a$

$\Gamma_{ji} = r = \phi_a n_b V \sigma$ (normalized wave function)

and $n_a = n_b = 1/V$

↳ Fermi's golden rule is defined for a single wave function describing a single particle state $\Rightarrow n = 1/V$

$r = \Gamma_{ji} = \dfrac{(v_a+v_b)\sigma}{V}$

$$\Rightarrow \sigma = \frac{\Gamma_{ji} V}{v_a+v_b}$$

show that Γ_{ji} scales as V^{-1} so that V drops out

$\Gamma_{ji} = \int_V \psi_1^* \psi_2^* H' \psi_a \psi_b$

$\sim \int_V \frac{1}{\sqrt{V}}\frac{1}{\sqrt{V}} H' \frac{1}{\sqrt{V}}\frac{1}{\sqrt{V}} \sim \int_V \frac{1}{V^2} \sim \frac{1}{V}$

so now we can safely put $V=1$

$$\hookrightarrow \sigma = \frac{\Gamma_{ji}'}{v_a+v_b} = \frac{(2\pi)^4\hbar^2}{v_a+v_b}\int |T_{ji}|^2 \delta(E_a+E_b-E_1-E_2)\delta^3(\vec{p}_a+\vec{p}_b-\vec{p}_1-\vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi\hbar)^3}\frac{d^3\vec{p}_2}{(2\pi\hbar)^3}$$

→ relativistic: (ψ' instead of ψ)

$$\sigma = \frac{c^4(2\pi)^{-2}\hbar^{-4}}{4E_aE_b(v_a+v_b)}\int |M_{ji}|^2 \delta(E_a+E_b-E_1-E_2)\delta^3(\vec{p}_a+\vec{p}_b-\vec{p}_1-\vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi\hbar)^3}\frac{d^3\vec{p}_2}{(2\pi\hbar)^3}$$

$$= \frac{(2\pi)^{-2}\hbar^{-4}c^4}{4E_aE_b(v_a+v_b)}\int |M_{ji}|^2\delta^4(p_a+p_b-p_1-p_2)\delta(p_1^2-m_1^2c^2)\Theta(p_1^0)\delta(p_2^2-m_2^2c^2)\Theta(p_2^0)d^4p_1 d^4p_2$$

expect Lorentz-invariant expression since the cross section can be seen as the interaction radius perpendicular to the direction of motion
→ no Lorentz contraction

* integral: Lorentz invariant (see last lecture)
* pre-factor: $4E_aE_b(v_a+v_b) = 4c^3\sqrt{(p_a\cdot p_b)^2 - m_a^2c^2m_b^2c^2}$

→ manifestly Lorentz invariant

check this we know that $\sqrt{(p_a\cdot p_b)^2 - m_a^2c^2m_b^2c^2}$ is manifestly Lorentz invariant $[(p_a\cdot p_b)^2$

show the equality

$\vec{p}_a, \vec{p}_b$ collinear and opposite

$(p_a\cdot p_b)^2 - m_a^2c^2m_b^2c^2 = \left(\frac{E_aE_b}{c^2} - \vec{p}_a\vec{p}_b\right)^2 - m_a^2c^2m_b^2c^2$

$= \frac{E_a^2E_b^2}{c^4} + 2p_ap_b\frac{E_aE_b}{c^2} + p_a^2p_b^2 - m_a^2c^2m_b^2c^2$

$= \frac{E_a^2E_b^2}{c^4} + \left(\frac{E_a^2}{c^2} - m_a^2c^2\right)\left(\frac{E_b^2}{c^2} - m_b^2c^2\right) - m_a^2c^2m_b^2c^2 + 2p_ap_b\frac{E_aE_b}{c^2}$

$= 2\frac{E_a^2E_b^2}{c^4} + 2p_ap_b\frac{E_aE_b}{c^2} - E_a^2m_b^2 - E_b^2m_a^2$

$\vec{p} = \gamma m\vec{v}$

$\gamma m = E/c^2$

since $p^0 = \gamma mc = E/c$

$\to \vec{p} = \frac{\vec{v}E}{c^2}$

$E_a^2E_b^2(v_a+v_b)^2 = E_a^2E_b^2(p_a/E_a + p_b/E_b)^2c^4$

$\vec{p} = \frac{\vec{v}E}{c^2} \to E = c^2\frac{\vec{p}}{\vec{v}}$

$\to \vec{v} = \frac{\vec{p}c^2}{E}$

$= c^4(p_aE_b + p_bE_a)^2$

$= c^4(p_a^2E_b^2 + p_b^2E_a^2 + 2p_ap_bE_aE_b)$

$= c^4\left[\left(\frac{E_a^2}{c^2} - m_a^2c^2\right)E_b^2 + \left(\frac{E_b^2}{c^2} - m_b^2c^2\right)E_a^2 + 2p_ap_bE_aE_b\right]$

$= c^6\left[2\frac{E_a^2E_b^2}{c^4} - m_a^2E_b^2 - m_b^2E_a^2 + 2p_ap_bE_aE_b\right]$

$= c^6\left[(p_a\cdot p_b)^2 - m_a^2c^2m_b^2c^2\right]$ → OK!

$$\Rightarrow \sigma = \frac{(2\pi)^{-2}\hbar^{-4}c^{-2}}{4\sqrt{(p_a\cdot p_b)^2 - m_a^2c^2m_b^2c^2}}\int |M_{ji}|^2\delta^4(p_a+p_b-p_1-p_2)\delta(p_1^2-m_1^2c^2)\Theta(p_1^0)\delta(p_2^2-m_2^2c^2)\Theta(p_2^0)d^4p_1d^4p_2$$

centre-of-mass scattering

since σ is Lorentz invariant, we can compute it in any frame
→ centre-of-mass frame: (CM-frame) → $p_a + p_b = p_1 - p_2$

$\vec{p}_a = -\vec{p}_b = \vec{p}_i \qquad \vec{p}_1 = -\vec{p}_2 = \vec{p}_f$

using $\vec{p} = \gamma m \vec{v} = E/c^2\, \vec{v}$

we can write

$$4E_aE_b(v_a+v_b) = 4E_aE_bc^2\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right) = 4c^2|\vec{p}_i|(E_a+E_b)$$

$$\Rightarrow \sigma = \frac{(2\pi)^{-2}\hbar^{-4}c^2}{4(E_a+E_b)|\vec{p}_i|}\int |M_{fi}|^2\delta(E_a+E_b-E_1-E_2)\delta^3(\vec{p}_1+\vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

$$= \frac{1}{64\pi^2\hbar^4}\frac{|\vec{p}_f|}{|\vec{p}_i|(E_a+E_b)^2}\int |M_{fi}|^2 d\Omega$$ ↙ same integral as for decay rate

check this

$$\sigma = \frac{(2\pi)^{-2}\hbar^{-4}c^4}{4E_aE_b(v_a+v_b)}\int |M_{fi}|^2\delta(E_a+E_b-E_1-E_2)\delta^3(\vec{p}_a+\vec{p}_b-\vec{p}_1-\vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

assume $\vec{p}_a = \vec{p}_b = 0$ (@ rest)
$p_a^0 = m_ac$
$p_b^0 = m_bc$

$$= \frac{(2\pi)^{-2}\hbar^{-4}c^2}{4|\vec{p}_i|(E_a+E_b)}\int |M_{fi}|^2\delta(E_a+E_b-E_1-E_2)\delta^3(\vec{p}_1+\vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$ ↙ cf. calc. p^* decay

$$= \frac{(2\pi)^{-2}\hbar^{-4}c^2}{4|\vec{p}_i|(E_a+E_b)}\frac{1}{4c^3}\int |M_{fi}|^2\delta(m_ac^2+m_bc^2-uc)\frac{\kappa(u)}{u}du\,d\Omega$$

$$= \frac{(2\pi)^{-2}\hbar^{-4}}{16|\vec{p}_i|(E_a+E_b)}\frac{\kappa(u=m_ac^2+m_bc^2)}{m_ac^2+m_bc^2}\int |M_{fi}|^2 d\Omega$$ ↙ $p^* = \kappa(u=E_a+E_b) = |\vec{p}_f|$ → size outgoing momentum

$$= \frac{1}{64\pi^2\hbar^4}\frac{|\vec{p}_f|}{|\vec{p}_i|(E_a+E_b)^2}\int |M_{fi}|^2 d\Omega$$

or $D = \dfrac{d\sigma}{d\Omega} = \dfrac{1}{64\pi^2\hbar^4}\dfrac{|\vec{p}_f|}{|\vec{p}_i|(E_a+E_b)^2}|M_{fi}|^2$

S-matrix and Feynman diagrams

- S-matrix
  - $\Psi_f = S\Psi_i$ operator that maps initial state to final state
  - since preservation of prob.: S is unitary $S^\dagger S = SS^\dagger = \mathbb{1}$
  - now $S = \lim_{t\to\infty}\exp\left(\frac{iHt}{\hbar}\right)$
- scattering through weak interaction
  - natural expansion S-matrix: Dyson expansion
  - in RQM: coincides with $T_{fi} = \langle f|H'|i\rangle + \sum_{j\neq i}\dfrac{\langle f|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j} + \dots$
- Feynman diagrams
  - = pictures that symbolise the various terms in the Dyson expansion
    → each term of M_{fi} (calc. perturbatively) is associated with a Feynman diagr.
  - * Feynman rules: which diagrams allowed and how we exactly associate a math. expression to each diagram
    → resulting series expansion (if param. small: asympt. series)
  - * QFT: all interactions proceed via particle interchange instead of interaction through some external potential
    external potential would imply action @ distance, however spec. rel. tells us that nothing can go faster than the speed of light
    $\Rightarrow T_{fi} = \underbrace{\langle f|H'|i\rangle}_{=0} + \sum_{j\neq i}\langle f|H'|j\rangle\langle j|H'|i\rangle(E_i-E_j)^{-1} + \dots$
    ↳ interaction through part./state exchange ↳ exchanged state $|j\rangle$
    ↳ would correspond to scattering in a potential

a simple toy-theory

– 2 particle scattering

$a + b \rightarrow c + d$ with interaction carried out by exchange of a boson force carrier X

→ 2 possible interactions

1. first $a \rightarrow X + c$, then $b + X \rightarrow d$

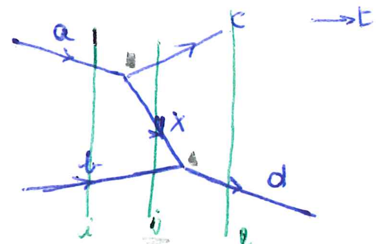
   ⇒ a emits the force carrier, b absorbs it

2. first $b \rightarrow X + d$, then $a + X \rightarrow c$

the total amplitude of this process will be the sum of these two time-ordered processes

→ consider 1. $T_{ji}^{(1)} = \dfrac{\langle f|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j}$

$$= \frac{\langle d|H'|X+b\rangle\langle X+c|H'|a\rangle}{(E_a+E_b) - (E_c+E_b+E_X)}$$



$|j\rangle = |X+b+c\rangle$ but interaction takes place at vertices and @ a c doesn't take part and @ d doesn't take part

now

$$\langle X+c|H'|a\rangle = \frac{M_{a\rightarrow X+c}}{\sqrt{\frac{2E_a}{c}\frac{2E_c}{c}\frac{2E_X}{c}}}$$

$$\langle d|H'|X+b\rangle = \frac{M_{X+b\rightarrow d}}{\sqrt{\frac{2E_X}{c}\frac{2E_b}{c}\frac{2E_d}{c}}}$$

assume matrix elements M to be constants

$M_{a\rightarrow X+c} = g_a$, $M_{X+b\rightarrow d} = g_b$

evaluating

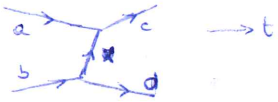
$$M_{ji}^{(1)} = T_{ji}^{(1)}\sqrt{\frac{2E_a}{c}\frac{2E_b}{c}\frac{2E_c}{c}\frac{2E_d}{c}}$$

$$T_{ji}^{(1)} = \frac{g_a g_b}{\frac{2E_X}{c}\sqrt{\frac{2E_a}{c}\frac{2E_c}{c}\frac{2E_b}{c}\frac{2E_d}{c}}} \times \frac{1}{E_a - E_c - E_X}$$

$$\Rightarrow M_{ji}^{(1)} = \frac{c\,g_a g_b}{2E_X(E_a - E_c - E_X)}$$

consider 2. analogously we find

$$M_{ji}^{(2)} = \frac{c\,g_a g_b}{2E_X(E_b - E_d - E_X)}$$



check $\langle X+d|H'|b\rangle = \dfrac{M_{b\rightarrow X+d}}{\sqrt{2E_b/c\;2E_d/c\;2E_X/c}} \approx \dfrac{g_b}{\sqrt{\dots}}$

$\langle c|H'|a+X\rangle = \dfrac{M_{a+X\rightarrow c}}{\sqrt{2E_a/c\;2E_c/c\;2E_X/c}} \approx \dfrac{g_a}{\sqrt{\dots}}$

$$\Rightarrow M_{ji}^{(2)} = \frac{g_a g_b}{\frac{2E_X}{c}(E_b - E_d - E_X)}$$

$E_i - E_j = (E_a + E_b) - (E_d + E_a + E_X)$

→ summation

+ use energy conservation: $E_a + E_b = E_c + E_d \rightarrow E_b - E_d = E_c - E_a$

$$M_{ji} = \frac{c\,g_a g_b}{(E_a - E_c)^2 - E_X^2}$$

with $E_X^2 = c^2[(\vec{p}_a - \vec{p}_c)^2 + m_X c^2]$ for both processes

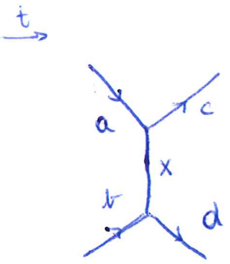
check $M_{ji} = M_{ji}^{(1)} + M_{ji}^{(2)} = \dfrac{c\,g_a g_b[(E_a - E_c - E_X) + (E_b - E_d - E_X)]}{2E_X(E_a - E_c - E_X)(E_b - E_d - E_X)}$

$$= \frac{c\,g_a g_b(-2E_X)}{2E_X(E_aE_b - E_cE_b - E_XE_b - E_aE_d - E_aE_X + E_cE_d + E_cE_X + E_XE_d + E_X^2)}$$

$$= \frac{c\,g_a g_b}{-[E_a(E_b - E_d - E_X) - E_c(E_b - E_d - E_X) - E_X(E_b - E_d - E_X)]}$$

$$= \frac{c\,g_a g_b}{(E_a - E_c + E_X)(E_a - E_c - E_X)} = \frac{c\,g_a g_b}{(E_a - E_c)^2 - E_X^2}$$

and $E_X^2 = c^2[(\vec{p}_a - \vec{p}_c)^2 + m_X c^2] = c^2[(\vec{p}_b - \vec{p}_d)^2 + m_X c^2]$ due to conservation of 3-mom and $(\vec{p}_b - \vec{p}_d)^2 = (\vec{p}_d - \vec{p}_b)^2$

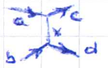


$$\Rightarrow M_{ji} = \frac{g_a g_b}{c[(p_a - p_c)^2 - m_X^2 c^2]} \equiv \frac{g_a g_b}{c[q^2 - m_X^2 c^2]}$$

with $q = p_a - p_c$ the 4-momentum of the virtual force carrier X

- 2 particle scattering

$$M_{fi} = \frac{q_a q_b / c}{q^2 - m_x^2 c^2}$$



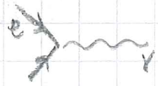
from this computation, we can learn that

* the sum of the prob. amplitudes associated to the separate time-ordered processes led to a Lorentz-invariant expression
* at the vertices of the time-ordered diagram energy is not conserved, but this violation only takes place in a (small) time interval since time-ordered diagrams themselves preserve E
* if the 2 diagrams are symbolized by a single diagram, then this can be seen as the exchange of a virtual particle.
  virtual: not on-shell → $q^2 \neq m_x^2 c^2$
  → necessary to preserve mom. & energy @ the separate vertices
  ↳ if it wouldn't be virtual: $M_{fi} \to \infty$ impossible!
* summing the 2 time-ordered processes → diagram for which both energy and mom. are preserved @ each vertex and for which M_{fi} is Lorentz invar at the expense of having a virtual particle

Check that at the vertices of the time-ordered diagram energy is not conserved

I don't know if this was what we needed to show →

example



→ impossible since you can make a restframe for $e^+ + e^-$ but there does not exist a restframe for a single photon (if $p_{photon} = 0$ → no photon exists)

Feynman diagrams: construct to write the contributions to M_{fi} at each other order expansion param (q_a, g_s) → # of vertices

towards a derivation of Feynman rules in QED

- interaction Hamiltonian H' for QED

from gauging principle $\partial_\mu \to \partial_\mu + iq_e A_\mu$ with q_e electric charge

we get Dirac eq. $i\hbar\gamma^\mu(\partial_\mu + iq_e A_\mu)\Psi = mc\Psi$

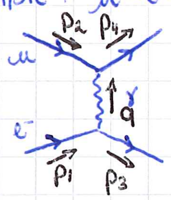
$\Rightarrow i\hbar\gamma^0\partial_0\Psi = (-i\hbar\gamma^i\partial_i + mc + q_e\hbar\gamma^\mu A_\mu)\Psi$

$\Rightarrow i\hbar\partial_t\Psi = \gamma^0(-i\hbar c\gamma^i\partial_i + mc^2 + q_e\hbar c\gamma^\mu A_\mu)\Psi \equiv H\Psi$

⇒ interaction Hamiltonian $H' = q_e\hbar c\gamma^0\gamma^\mu A_\mu$

- Feynman diagrams

example: μ^--e^--scattering

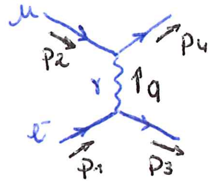


2 processes
1. μ sends out γ which is captured by the e⁻
2.

scattering between 2 (diff.) fermions
* sum 2 time-ordered processes
* 2 interaction vertices with interaction by (virtual) photon
* ⇒! photon has 2 polarization states → need to sum over them

! matrix element interaction vertex not a simple constant ($\langle\Psi_f|H'|\Psi_i\rangle \neq$ cte)

- electron-muon scattering



* at e^- vertex: $\langle e^-(p_3)|H'|e^-(p_1)\rangle$

$$= \underbrace{q_e \hbar c}_{H' = q_e\hbar c\gamma^0\gamma^\mu A_\mu} \exp\left(\tfrac{-i}{\hbar} q_\nu x^\nu\right) u_e^+(p_3)\gamma^0\gamma^\mu \varepsilon_\mu^{(s)} u_e(p_1)$$

where we've used that
$$A^\mu = (\varepsilon^\mu)^{(s)} \exp\left(\tfrac{-i}{\hbar} q_\mu x^\mu\right)$$
since the photon is represented by the plane wave (since we consider eigenmom. states) with s the unspecified polarization state

* at μ vertex: $\langle \mu(p_4)|H'|\mu(p_2)\rangle$

$$= q_m \hbar c \exp\left(\tfrac{i}{\hbar} q_\nu x^\nu\right) u_m^+(p_4)\gamma^0\gamma^\lambda (\varepsilon_\lambda^{(s)})^* u_M(p_2)$$

where the photon is now conjugated

explain we create the photon at 1 vertex and we absorb it at another → take the conjugate (otherwise we would create 2 photons)

$$\Rightarrow M = q_e q_m \hbar^2 c^2 \sum_s \left[u_e^+(p_3)\gamma^0\gamma^\mu u_e(p_1) \frac{\varepsilon_\mu^{(s)}(\varepsilon_\lambda^{(s)})^*}{q^2} \cdot u_m^+(p_4)\gamma^0\gamma^\mu u_m(p_2)\right]$$

where we've summed over the 2 possible time-orderings, creating the term $\frac{1}{q^2 - m_x^2 c^2}$ where now $m_x = 0$ ($x = \gamma \to$ photon)

explain why we sum the M's for every spin and not the $|M|^2$

It is possible to have phase factors for your spins and you don't want them to vanish

now $\sum_s \varepsilon_\mu^{(s)}(\varepsilon_\lambda^{(s)})^* = -\eta_{\mu\lambda}$ (see later)

$$\Rightarrow M = -q_e q_M \hbar^2 c^2\, \bar{u}_e(p_3)\gamma^\mu u_e(p_1) \frac{\eta_{\mu\lambda}}{q^2} \bar{u}_M(p_4)\gamma^\mu u_M(p_2)$$

where $\bar{u} = u^+\gamma^0$

which is manifest Lorentz invariant

- proof $\sum_s \varepsilon_\mu^{(s)}(\varepsilon_\lambda^{(s)})^* = -\eta_{\mu\lambda}$

at the vertices of the time-ordered diagrams (photons on their mass-shell, non virtual)
$\langle j|H'|i\rangle$ or $\langle j|H'|j\rangle$ can be written as $j_\mu \varepsilon^\mu$ or the complex conj.
where j_μ is the e^- or μ current ($\bar{\Psi}(p_3)\gamma^\mu\Psi(p_1)$)

insisting that our theory is gauge-invariant, our matrix elements should not change under gauge transfo
→ Lorentz gauge: $A_\mu \to A_\mu + \partial_\mu \lambda$

$\Rightarrow \Box\lambda = 0$

$\Rightarrow \lambda = a\exp(ip\cdot x)$ with p the on-shell photon mom.: $p^2 = 0$

$\Rightarrow \varepsilon_\mu \to \varepsilon_\mu - a i p_\mu$ for any a

explain the gauge transfo doesn't fix everything there is still 1 remaining freedom adding this change does fix everything

→ matrix elements can only be gauge invariant if
$$j_\mu p^\mu = 0$$

↳ if we write Feynman diagram as sum of 2 time-ordered diagrams
→ exchanged photon on mass shell
→ can use $j_\mu p^\mu = 0$

$\Rightarrow \sum_s (\varepsilon^\mu)^* \varepsilon^\nu$ can be written in Coulomb gauge as

$\sum_s (\varepsilon^i)^* \varepsilon^j = \delta^{ij} - \hat{p}^i \hat{p}^j$ and since $j_\mu p^\mu = 0 \rightarrow j_i \hat{p}^i = -j_0 \frac{p^0}{|\vec{p}|} = \pm j_0$

$$\Rightarrow j_\mu j_\nu (\varepsilon^\mu)^* \varepsilon^\nu = j_i j_j (\delta_{ij} - \hat{p}^i \hat{p}^j)$$
$$= j_i j_j \delta_{ij} - j_0^2$$
$$= \eta^{\mu\nu} j_\mu j_\nu$$

$\Rightarrow$ ~~photons~~ polarisations $\perp$

Feynman rules

- = algorithm that allows you to calculate a contribution to $\mathcal{M}$ for a given diagram - no need to derive the matrix elements for each process
- → you'll find that observables are given by a perturbative expansion in the dimless fine-structure constant $\alpha = e^2/\hbar c \sim 1/137$ ~ divergent, thought of as an asymptotic series

* an extra rule:
Feynman diagrams are build from simple vertex diagrams that can be glued together in all possible ways you would like
→ e.g. in QED all diagrams build from 

6 Electron-muon scattering

how to obtain the actual cross section to leading order in α
→ will hold for all scattering betw. different fermions
↳ computing $\mathcal{M}$ explicitly
say $q_e = q_m = 1$ and $c = \hbar = 1$ and $p_1, p_2, p_3, p_4 \to 1, 2, 3, 4$

$$\Rightarrow \quad \mathcal{M} = -\frac{e^2}{q^2}\, \bar{u}(3)\gamma^\mu u(1)\eta_{\mu\lambda}\, \bar{u}(4)\gamma^\lambda u(2)$$

Summing over the spins

* Casimir's trick for amplitudes/probabilities
  - average over the initial spins ← you do not measure the initial spins
  - sum over the outgoing spins ← scattering event is measured and adds to the amplitude of the process

since typically in an experiment:
the beam of part. with mixed spin-up and spin-down is of equal amount and and after interaction the particles are treated the same indep. of their spin

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \langle(\mathcal{M}\mathcal{M}^*)\rangle$$

$$= \left(\frac{e^2}{q^2}\right)^2 \frac{1}{4} \sum_{s_1,s_2,s_3,s_4} |\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)\eta_{\mu\lambda}\, \bar{u}^{s_4}(4)\gamma^\lambda u^{s_2}(2)|^2$$

↳ from averaging over 4 initial spins (2 for each ingoing part)

$$= \frac{e^4}{q^4}\frac{1}{4}\sum_{s_1,s_2,s_3,s_4} [\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)][\bar{u}^{s_4}(4)\gamma_\mu u^{s_2}(2)] \cdot [\bar{u}^{s_3}(3)\gamma^\nu u^{s_1}(1)]^*[\bar{u}^{s_4}(4)\gamma_\nu u^{s_2}(2)]^*$$

check how the rewriting happened

$$|\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)\eta_{\mu\lambda}\, \bar{u}^{s_4}(4)\gamma^\lambda u^{s_2}(2)|^2 = (\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)\eta_{\mu\lambda}\,\bar{u}^{s_4}(4)\gamma^\lambda u^{s_2}(2))(\bar{u}^{s_3}(3)\gamma^\nu u^{s_1}(1)\eta_{\nu\sigma}\,\bar{u}^{s_4}(4)\gamma^\sigma u^{s_2}(2))^*$$

$x_\mu = \eta_{\mu\nu}x^\nu$ ↳

$$= [\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)\; \bar{u}^{s_4}(4)\gamma_\mu u^{s_2}(2)][\bar{u}^{s_3}(3)\gamma^\nu u^{s_1}(1)\; \bar{u}^{s_4}(4)\gamma_\nu u^{s_2}(2)]^*$$

* for computing the complex conjugate:
  a spinor bilinear $\bar{u}(a)\Gamma u(b)$ with Γ some matrix
  $\to (\bar{u}(a)\Gamma u(b))^* = \bar{u}(b)\bar{\Gamma}u(a)$ where $\bar{\Gamma} = \gamma^0\Gamma^+\gamma^0$

  prove this

$$(\bar{u}(a)\Gamma u(b))^* = (\bar{u}(a)\Gamma u(b))^+ = u^+(b)\Gamma^+(u^+(a)\gamma^0)^+$$
$$= u^+(b)\Gamma^+(\gamma^0)^+u(a) = u^+(b)\gamma^0\gamma^0\Gamma^+\gamma^0 u(a)$$
$$= \bar{u}(b)\bar{\Gamma}u(a)$$

↗ $\gamma^0\gamma^0 = \mathbb{1}$, $(\gamma^0)^+ = \gamma^0$, $\bar{u} = u^+\gamma^0$

and since $\bar{\gamma}^\mu = \gamma^\mu$
  prove this see proof in section "transforming Dirac spinors" (ch 3)

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \frac{e^4}{q^4}\frac{1}{4}\sum_{s_1,s_3}[\bar{u}^{s_3}(3)\gamma^\mu u^{s_1}(1)][\bar{u}^{s_1}(1)\gamma^\nu u^{s_3}(3)] \cdot \sum_{s_2,s_4}[\bar{u}^{s_4}(4)\gamma_\mu u^{s_2}(2)][\bar{u}^{s_2}(2)\gamma_\nu u^{s_4}(4)]$$

Casimir:
consider a general product of ~~spnt~~ spinor bilinears G
$G = \bar{u}(a)\Gamma_1 u(b)\bar{u}(b)\Gamma_2 u(a)$
→ sum over the b-spins: $\sum_{b\text{-spins}} G = \bar{u}(a)\Gamma_1\left(\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b)\right)\Gamma_2 u(a)$

$$= \bar{u}(a)\Gamma_1(\gamma^\mu p^b_\mu + m_b\mathbb{1})\Gamma_2 u(a)$$

→ sum over the a-spins (see back of this page) call $Q = \Gamma_1 (\gamma^\mu p^b_\mu + m_b)\Gamma_2$

$$\to \sum_{\substack{a\text{-spins}\\ b\text{-spins}}} G = \sum_{s_a} \bar{u}(a)^{s_a} Q\, u^{s_a}(a)$$

$$= \sum_{s_a} \sum_{\alpha,\beta} \bar{u}(a)^{s_a}_\alpha Q_{\alpha\beta} u(a)^{s_a}_\beta$$

$$= \sum_{\alpha,\beta} \sum_{s_a} u(a)^{s_a}_\beta \bar{u}(a)^{s_a}_\alpha Q_{\alpha\beta}$$

$$= \sum_{\alpha,\beta} (\gamma^\mu p^a_\mu + m_a \mathbb{1})_{\beta\alpha} Q_{\alpha\beta}$$

$$= \mathrm{Tr}\left[(\gamma^\mu p^a_\mu + m_a \mathbb{1})\Gamma_1 (\gamma^\mu p^b_\mu + m_b)\Gamma_2\right]$$

Making traces

applying Casimir's trick twice, we find

$$\langle|M|^2\rangle = \frac{e^4}{4(p_1-p_3)^4} \mathrm{Tr}[\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_3+m)] \times \mathrm{Tr}[\gamma_\mu(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)]$$

where $\not{p} = \gamma^\mu p_\mu$ and we used conservation of mom. $q^2 = (p_1-p_3)^2$

$$\hookrightarrow \mathrm{Tr}[\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_3+m)] = \mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3 + m\gamma^\mu\gamma^\nu\not{p}_1 + m\gamma^\mu\gamma^\nu\not{p}_3 + m^2\gamma^\mu\gamma^\nu]$$

* $\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3] \to$ trace of 4 γ-matrices

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda})$$

prove this

using anti-commutation relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = \mathrm{Tr}[2\eta^{\mu\nu}\gamma^\lambda\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma]$$

$$= \mathrm{Tr}[2\eta^{\mu\nu}\gamma^\lambda\gamma^\sigma] - \mathrm{Tr}[2\eta^{\mu\lambda}\gamma^\nu\gamma^\sigma] + \mathrm{Tr}[\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma]$$

$\mathrm{Tr}[\gamma^\lambda\gamma^\sigma] = 4\eta^{\lambda\sigma}$ (see later)

$$= 8\eta^{\mu\nu}\eta^{\lambda\sigma} - 8\eta^{\mu\lambda}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\nu\lambda} - \mathrm{Tr}[\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu]$$

and since the Trace is invariant under cyclic permutations, we get

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda})$$

* $\mathrm{Tr}[m\gamma^\mu\gamma^\nu\not{p}] \to$ trace of 3 γ-matrices

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\sigma] = 0 \qquad \text{(see appendix E)}$$

* $\mathrm{Tr}[m^2\gamma^\mu\gamma^\nu] \to$ trace of 2 γ-matrices

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$$

prove this

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \mathrm{Tr}[2\eta^{\mu\nu}\mathbb{1}] - \mathrm{Tr}[\gamma^\nu\gamma^\mu]$$

$$\Rightarrow 2\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 2\eta^{\mu\nu}\mathrm{Tr}[\mathbb{1}]$$

$$\Rightarrow \mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$$

?

$$\to \mathrm{Tr}[\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_3+m)] = 4[\eta^{\mu\nu}(m^2 - p_1\cdot p_3) + \overbrace{p_1^\mu p_3^\nu + p_3^\mu p_1^\nu}]$$

$$\Rightarrow \langle|M|^2\rangle = \frac{4e^4}{(p_1-p_3)^4}\{p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + \eta^{\mu\nu}(m^2 - p_1\cdot p_3)\} \times \{(p_2)_\mu(p_4)_\nu + (p_4)_\mu(p_2)_\nu + \eta_{\mu\nu}(M^2 - p_2\cdot p_4)\}$$

$$= \frac{8e^4}{(p_1-p_3)^4}\{(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)M^2 - (p_2\cdot p_4)m^2 + 2m^2M^2\}$$

$\mathrm{Tr}(\eta^{\mu\nu}\eta_{\mu\nu}) = 4$

pelo!

Non-relativistic limit

- differential cross-section in lab frame

$(\hbar = c = 1)$ $\quad D = \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_f|}{|\vec{p}_i|(E_a + E_b)^2} |\mathcal{M}_{fi}|^2$

now our collision is elastic $\Rightarrow |\vec{p}_i| = |\vec{p}_f|$

explain why (not sure about my answer)

because the particles coming out of the interaction are the same as the incomming ones and the interaction happens with the help of a photon $(m_\gamma = 0)$ $\rightarrow$ no mass exchange

and we can ignore the recoil of the muon $(m_{muon} >> m_{e^-})$

$\Rightarrow$ ZM ~~frame~~ / CM-frame coincide with lab frame

and $\quad E_{muon} = M \geq E_{el} \quad (M = m_{muon})$

why - why what?

no recoil & lab frame $\rightarrow$ μ doesn't move $\rightarrow$ E_μ = rest energy μ with $c=1$

$E_\mu > E_{el}$ since idle

$\Rightarrow D = \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \langle |\mathcal{M}|^2 \rangle \frac{1}{M^2}$

concretely the momenta can be taken as follows

$p_1 = (E_{el}, \vec{p}_1)$, $p_2 = (M, \vec{0})$, $p_3(E_{el}, \vec{p}_3)$ $\quad p_4(M, \vec{0})$

where we've used that recoil is negligible and the interaction is elastic

angle between $\vec{p}_1$ and $\vec{p}$ = scattering angle θ

$\rightarrow \vec{p}_1 \cdot \vec{p}_3 = |\vec{p}|^2 \cos\theta$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle = \left(\frac{e^2 M}{|\vec{p}|^2 \sin^2(\theta/2)} \right)^2 [m^2 + |\vec{p}|^2 \cos^2(\theta/2)]$$

check this

$\langle |\mathcal{M}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \{ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M^2 - (p_2 \cdot p_4)m^2 + 2m^2M^2 \}$

$p_1 = (E_{el}, \vec{p}_1)$, $p_2 = (M, \vec{0})$, $p_3 = (E_{el}, \vec{p}_3)$, $p_4 = (M, \vec{0})$

$(p_1 - p_3) =$

$(p_1 \cdot p_2) = E_{el} \cdot M + 0 = E_{el} \cdot M$

$(p_3 \cdot p_4) = E_{el} \cdot M$

$(p_1 \cdot p_4) = E_{el} \cdot M$

$(p_2 \cdot p_3) = E_{el} \cdot M$

$(p_1 \cdot p_3) = E_{el}^2 - \vec{p}_1 \cdot \vec{p}_3 = E_{el}^2 - |\vec{p}|^2 \cos\theta$

$(p_2 \cdot p_4) = M^2$

$\Rightarrow \langle |\mathcal{M}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \{ E_{el}^2 M^2 + E_{el}^2 M^2 - E_{el}^2 M^2 + |\vec{p}|^2 M^2 \cos\theta - M^2 m^2 + 2m^2 M^2 \}$

$= \frac{8e^4}{(p_1 - p_3)^4} \{ E_{el}^2 M^2 + |\vec{p}|^2 M^2 \cos\theta + m^2 M^2 \}$

↳ doesn't really add up yet

$\Rightarrow D = \left(\frac{e^2}{8\pi |\vec{p}|^2 \sin^2(\theta/2)} \right)^2 [m^2 + |\vec{p}|^2 \cos^2(\theta/2)]$ $\quad$ Mott formula

non-relativistic limit $\quad |\vec{p}|^2 / m^2 << 1 \quad (c=1)$

$\Rightarrow D = \left(\frac{e^2 m}{8\pi |\vec{p}|^2 \sin^2(\theta/2)} \right)^2$ $\rightarrow$ Rutherford's formula