Electrodynamics

H2: Electrostatics

Coulomb's Law: $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$ with $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$\vec{F} = Q\vec{E} \Rightarrow$ discreet: $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$; continuous: $\begin{cases} \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' \\ \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' \\ \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' \end{cases}$

flux of $\vec{E}$ through surface S: $\Phi_E = \int_S \vec{E}\cdot d\vec{a}$

Gauss's law: $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$; $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\left[\begin{matrix}\oint \vec{E}\cdot d\vec{l} = 0 \\ \vec{\nabla}\times\vec{E} = \vec{0}\end{matrix}\right]$

(H1: $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$; $\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\vec{\mathscr{r}})$)

infinite plane with uniform charge $\sigma \rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$; $\frac{\partial V}{\partial n} = \nabla V\cdot\hat{n}$; $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

$V(\vec{r}) = -\int_O^{\vec{r}} \vec{E}\cdot d\vec{l}'$; $\vec{E} = -\nabla V$; $V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$; $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

discreet: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{\mathscr{r}_i}$; continuous: $\begin{cases} V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}}dl' \\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}}da' \\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}d\tau' \end{cases}$

(pg 88 for summary)

Work to move a charge: $W = QV(\vec{r})$

work to set up point charges: $W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$; $W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$; $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$

energy of continuous charge distribution: $W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{a}\right) = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} E^2 d\tau = \frac{1}{2}\int \rho V d\tau$

Conductors
- $\vec{E} = \vec{0}$ inside
- equipotential
- $\rho = 0$ inside
- any net charge is on surface
- $\vec{E} \perp$ surface

surface charge on conductor
- immediately outside $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$; $\sigma = -\varepsilon_0\frac{\partial V}{\partial n}$
- force per unit are $\vec{f} = \sigma\vec{E}_{average} = \frac{\sigma^2}{2\varepsilon_0}\hat{n} = \frac{\sigma}{2}(\vec{E}_{above} + \vec{E}_{below})$
- electrostatic pressure $P = \frac{\varepsilon_0}{2}E^2$

capacitor: $C = \frac{Q}{V}$ with $V = V_{+Q} - V_{-Q}$

$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$ with V the final potential

field between 2 infinite parallel plates with $\pm\sigma \rightarrow E = \frac{\sigma}{\varepsilon_0}$

H3 Potentials

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ is Poisson's equation

We'll look where there's no charge $\to \rho = 0 \to \nabla^2 V = 0$ (Laplace's equation)

$\left.\begin{array}{l}\to V \text{ is the average of the points around it} \\ \to \text{no local maxima/minima}\end{array}\right\} \Rightarrow \begin{cases} 1D: V(x) = \frac{1}{2}\left(V(x-a) + V(x+a)\right) \\ 2D: V(x,y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl \\ 3D: V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da \end{cases}$

(uniqueness theorems → pg 119)

ground $\to V = 0$

method of images: $\begin{cases} * \text{ use charges with different sign} \\ * \text{ don't put those charges in the region where you're calculating } V \end{cases}$

separation of variables: $\begin{cases} V(x,y,z) = X(x)\,Y(y)\,Z(z) \\ V(r,\theta) = R(r)\,\Theta(\theta) \quad \text{(azimuthal symmetry)} \end{cases}$

$$\Downarrow$$

$$= \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta))$$

Legendre polynomials; $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \to \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$

$$= \int_0^{\pi} P_m(\cos\theta)\,P_n(\cos\theta)\sin\theta\,d\theta$$

multipole expansion: $\pm q$ at distance $d \to V(\vec{r}) \cong \frac{1}{4\pi\varepsilon_0}\frac{qd\cos(\theta)}{r^2}$ for $r \gg d$

charge distribution $\to V(\vec{r}) \cong \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos(\alpha))\,\rho(\vec{r}')\,d\tau'$

for $r \gg r'$

dipole moment: $\vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau' = \sum q_i \vec{r}_i'$

$\to V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

dipole field: $\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$, for $r \gg d$, $\vec{p} = p\hat{z}$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

MS magnetostatics

$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$; $\vec{F}_{tot} = Q[\vec{E} + \vec{v} \times \vec{B}]$

surface current density: $\begin{cases} \vec{K} = \dfrac{d\vec{I}}{dl_\perp} \\ \vec{K} = \sigma \vec{v} \\ \vec{F}_{mg} = \int (\vec{K} \times \vec{B})\, da \end{cases}$

cyclotron motion: $QvB = m\frac{v^2}{R} \Rightarrow p = QBR$

Magnetic forces do no work!

volume current density: $\begin{cases} \vec{J} = \dfrac{d\vec{I}}{da_\perp} \\ \vec{J} = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- \\ \vec{F}_{mg} = \int (\vec{J} \times \vec{B})\, d\tau \\ \vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t} \end{cases}$

magnetostatics $\rightarrow \vec{\nabla} \cdot \vec{J} = 0$

$\vec{F}_{mag} = \int I (d\vec{l} \times \vec{B})$

for $\vec{K} \times \hat{r}\, da'$; $\vec{J} \times \hat{r}\, d\tau'$

Biot-Savart Law: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{r}}{r^2}$

2 parallel infinite wires → force per unit length: $f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$

only for steady currents!

↳ infinite wire: $B = \frac{\mu_0 I}{2\pi s}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$; $I_{enc} = \int \vec{J} \cdot d\vec{a}$; $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$; $\vec{\nabla} \cdot \vec{B} = 0$

Ampère's Law

Magnetic vector potential: $\vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{\nabla} \cdot \vec{A} = 0$, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$; $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{r} d\tau'$

(pg 250 for summary)

$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$; $\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$

Magnetic flux: $\Phi_B = \int \vec{B} \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$

Multipole expansion: $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$

$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$ with $\vec{m} = I \int d\vec{a} = I\vec{a}$ (magnetic dipole moment)

"pure" dipole: $\vec{B}_{dip}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0 m}{4\pi r^3} (2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$, with $\vec{m} = m\hat{z}$

Toroidal coil: $\vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s} \hat{\phi} & \text{for points inside the coil} \\ \vec{0} & \text{for outside} \end{cases}$

↳ pg 239

For a configuration of charges and currents confined within a volume $\mathcal{V}$:

$\int_{\mathcal{V}} \vec{J}\, d\tau = \frac{d\vec{p}}{dt}$ with $\vec{p}$ = total dipole moment

H7 electrodynamics

$$\vec{J} = \underbrace{\sigma \vec{f}}_{} \quad ; \quad \vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \approx \sigma \vec{E} \leftarrow \text{Ohm's Law}$$

↳ conductivity (≠ surface charge) ↳ force per unit charge

resistivity: $\rho = \frac{1}{\sigma}$ Joule heating Law: $P = VI = I^2 R$ $\left[\frac{J}{s}\right]$ or [Watt]

Ohm's Law: $V = IR$

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} \quad \text{with } \vec{f} = \vec{f}_s + \vec{E}$$

↳ electromotive force (emf)

motional emf: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ with $\Phi_B = \int \vec{B} \cdot d\vec{a}$; $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$

A changing magnetic field induces an electric field!

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \quad ; \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Inductance: imagine 2 loops → $\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = M_{21} I_1$

with mutual inductance $M_{21} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}$

[Henry] selfinductance L: $\Phi = LI$; $\mathcal{E} = -L \frac{dI}{dt}$

↳ back emf

Work and energy: $W = \frac{1}{2} L I^2$

$$W = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} B^2 \, d\tau$$

non-steady currents: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

A chanching electric field induces a magnetic field!

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

H8 Conservation Laws

Conservation of charge: $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Total energy stored (per unit volume): $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$

Poynting's theorem: $\frac{dW}{dt} = \int_V (\vec{E} \cdot \vec{J}) d\tau = -\frac{d}{dt}\int_V u \, d\tau - \oint_S \vec{S} \cdot d\vec{a}$

Poynting vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is the energy flux density / energy per area per time of the field

→ if $\frac{dW}{dt} = 0 \rightarrow \frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S}$

↳ $\int_S \vec{S} \cdot d\vec{a}$ is the energy per unit time passing through this surface (Power)

Maxwell stress tensor: $T_{ij} = \varepsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$ with $i, j \in \{x, y, z\}$

→ force per unit volume $\vec{f} = \vec{\nabla} \cdot \overleftrightarrow{T} - \varepsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$

→ total force on the charges in V: $\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \varepsilon_0 \mu_0 \frac{d}{dt}\int_V \vec{S} \, d\tau = \frac{d\vec{p}_{mech}}{dt}$

Momentum density in the fields: $\vec{g} = \mu_0 \varepsilon_0 \vec{S} = \varepsilon_0 (\vec{E} \times \vec{B})$

→ if $\frac{d\vec{p}_{mech}}{dt} = 0 \rightarrow \frac{\partial \vec{g}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}$

Momentum stored in the fields: $\vec{p} = \int_V \vec{g} \, d\tau = \mu_0 \varepsilon_0 \int_V \vec{S} \, d\tau$

$\overleftrightarrow{T}$: electromagnetic stress (force per unit area) acting on a surface

$-\overleftrightarrow{T}$: describes the flow of momentum (it is the momentum current density) carried by the fields

Angular momentum density: $\vec{\ell} = \vec{r} \times \vec{g} = \varepsilon_0 [\vec{r} \times (\vec{E} \times \vec{B})]$

→ total angular momentum in fields: $\vec{L} = \int \vec{\ell} \, d\tau$

Intensity $I = |\langle \vec{S} \rangle|$

H9 Electromagnetic waves

(classical) wave equation: $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ with $v = \sqrt{\frac{T}{\mu}}$ → tension; → mass per unit length

→ solutions: $f(z,t) = g(z-vt) + h(z+vt)$

↳ moves in $\hat{z}$ ↳ moves in $-\hat{z}$

Sinusoidal wave: $f(z,t) = A\cos[k(z-vt)+\delta]$ → phase constant

↳ phase

↳ wave number

→ phase = 0 → $z = vt - \frac{\delta}{k}$

$\lambda = \frac{2\pi}{|k|}$; $T = \frac{2\pi}{|k|v}$; $f = \frac{1}{T} = \frac{v}{\lambda}$

$\omega = 2\pi f = |k|v$

→ wave in $\hat{z}$ direction: $f(z,t) = A\cos(kz-\omega t+\delta)$

→ wave in $-\hat{z}$ direction: $f(z,t) = A\cos(kz+\omega t-\delta) = A\cos(-kz-\omega t+\delta)$

reflection and transmission: see pg. 388

Polarization: waves traveling in $\hat{z}$ direction: $\vec{\tilde{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$, with $\hat{n}\cdot\hat{z} = 0$

→ $f(z,t) = R_e[\tilde{f}(z,t)]$; $\tilde{A} = Ae^{i\delta}$

Waves in vacuum: $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$; $\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$

→ monochromatic waves: $\begin{cases}\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 e^{i(kz-\omega t)} \\ \vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_0 e^{i(kz-\omega t)}\end{cases}$ with $\omega = ck$

↳ $(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$

↳ $\vec{\tilde{B}}_0 = \frac{k}{\omega}(\hat{z}\times\vec{\tilde{E}}_0)$ ⇒ $\vec{E}$ & $\vec{B}$ are in phase and mutually perpendicular; $B_0 = \frac{1}{c}E_0$

↳ In general: $\begin{cases}\vec{\tilde{E}}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{n} & \text{with } \hat{n}\cdot\hat{k} = 0 \\ \vec{\tilde{B}}(\vec{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{1}{c}\hat{k}\times\vec{\tilde{E}}\end{cases}$

→ monochromatic waves in $\hat{z}$ direction:

$\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0^2$

$\langle\vec{S}\rangle = \frac{1}{2}c\varepsilon_0 E_0^2\hat{z}$

$\langle\vec{g}\rangle = \frac{1}{2c}\varepsilon_0 E_0^2\hat{z}$

average power per unit area = intensity: $I = \langle S\rangle = \frac{1}{2}c\varepsilon_0 E_0^2$

perfect absorber: $\Delta\vec{p} = \langle\vec{g}\rangle Ac\Delta t$

→ radiation pressure $P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c}$

perfect reflector: → pressure is twice as great

EM waves in matter: see section 9.3 (no subject matter)

→ idem for 9.4 and 9.5

H10 Potentials and Fields

In electrodynamics $\vec{\nabla}\times\vec{E} = \frac{\partial \vec{B}}{\partial t} \neq \vec{0} \rightarrow \vec{E} \neq -\vec{\nabla}V$

$$\Rightarrow \vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

Gauge transformations (don't alter $\vec{E}$ or $\vec{B}$)

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\lambda \\ V' = V - \frac{\partial \lambda}{\partial t} \end{cases}$$

→ Coulomb Gauge: pick $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \nabla^2 V = -\frac{1}{\varepsilon_0}\rho \Rightarrow V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}}\,d\tau'$

with $V(\infty,t)=0$

* advantage: V simple
* disadvantage: $\vec{A}$ difficult → $\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t} = -\mu_0\vec{J} + \mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$

→ Lorenz Gauge: pick $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}$

$$\begin{cases} \Rightarrow \nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \\ \Rightarrow \nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho \end{cases}$$

take $\underbrace{\Box^2}_{\text{d'Alembertian}} \equiv \nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2} \Rightarrow \begin{cases} \Box^2 V = -\frac{1}{\varepsilon_0}\rho \\ \Box^2\vec{A} = -\mu_0\vec{J} \end{cases}$

Lorentz force law in potential form:

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) = q\left[-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right]$$

$$= -q\left[\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{\nabla}(V - \vec{v}\cdot\vec{A})\right]$$

convective derivative of $\vec{A}$: $\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$

canonical momentum: $\vec{p}_{can} = \vec{p} + q\vec{A}$

Retarded potentials: $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$; $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$

with $t_r = t - \frac{\mathscr{r}}{c}$

Jefimenko's Equations: $\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{\rho(\vec{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'$$

Moving point charge → Liénard-Wiechert potentials:

$$\begin{cases} V(\vec{r},t) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})} & \text{with } \vec{\mathscr{r}} \text{ and } \vec{v} \text{ at } t_r \\[2ex] \vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi} \dfrac{qc\vec{v}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})} = \dfrac{\vec{v}}{c^2} V(\vec{r},t) \end{cases}$$

$$|\vec{r} - \vec{w}(t_r)| = c(t-t_r)$$

$$\vec{\mathscr{r}} = \vec{r} - \underbrace{\vec{w}(t_r)}_{\text{position of } q}$$

$$\Rightarrow \begin{cases} \vec{E}(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0} \dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right] \\[2ex] \vec{B}(\vec{r},t) = \dfrac{1}{c} \hat{\mathscr{r}} \times \vec{E}(\vec{r},t) \end{cases}$$

with $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

$\vec{a} = \dot{\vec{v}} = \ddot{\vec{w}}(t_r)$

$\vec{\mathscr{r}} \times (\vec{u} \times \vec{a})$: radiation field or acceleration field $\sim \frac{1}{\mathscr{r}}$

$(c^2 - v^2)\vec{u}$: generalized Coulomb field or velocity field $\sim \frac{1}{\mathscr{r}^2}$

⇒ Lorentz force law:

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3} \left\{ \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right] + \frac{\vec{V}}{c} \times \left[\hat{\mathscr{r}} \times \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right] \right] \right\}$$

with $\vec{V}$ the velocity of Q, and $\{\vec{\mathscr{r}}, \vec{u}, \vec{v}, \vec{a}\}$ are evaluated at t_r

Charge moving at constant speed: see examples 10.3 and 10.4

H11 Radiation

Electric dipole radiation: $q(t) = q_0 \cos(\omega t)$ (see fig 11.2) and $d \ll \lambda \ll r$, with $\lambda = \frac{c}{\omega}$

$$\Rightarrow \begin{cases} V(r,\theta,t) = -\frac{p_0 \omega}{4\pi\varepsilon_0 c}\left(\frac{\cos(\theta)}{r}\right)\sin\left(\omega(t - r/c)\right) \\ \vec{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left(\omega(t - r/c)\right)\hat{z} \\ \vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin(\theta)}{r}\right)\cos\left(\omega(t - r/c)\right)\hat{\theta} \\ \vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin(\theta)}{r}\right)\cos\left(\omega(t - r/c)\right)\hat{\phi} \end{cases}$$

$\Rightarrow$ monochromatic waves of frequency ω in radial direction and $\frac{E_0}{B_0} = c$

$$\langle P \rangle = \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} = \text{total power radiated}$$

Magnetic dipole radiation: $I(t) = I_0 \cos(\omega t)$ (see fig 11.8) and $b \ll \lambda \ll r$, with $\lambda = \frac{c}{\omega}$

$$\Rightarrow \begin{cases} \vec{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left(\omega(t - r/c)\right)\hat{\phi} \\ \vec{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega(t - r/c)\right)\hat{\phi} \\ \vec{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega(t - r/c)\right)\hat{\theta} \end{cases}$$

$$\langle P \rangle = \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

$$\Rightarrow \frac{P_{mag}}{P_{elec}} = \left(\frac{m_0}{p_0 c}\right)^2 \underset{\text{if } d = \pi b}{=} \left(\frac{\omega b}{c}\right)^2 \ll 1$$

Radiation from an arbitrary source

$$\vec{E}(\vec{r},t) \cong \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}\right] = \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right] \quad \text{with } \ddot{\vec{p}} = \ddot{\vec{p}}(t_0) = \ddot{\vec{p}}\left(t - \frac{r}{c}\right)$$

($\vec{p}$: dipole moment)

$$\vec{B}(\vec{r},t) \cong -\frac{\mu_0}{4\pi r c}\left[\hat{r}\times\ddot{\vec{p}}\right]$$

choose z axis in the direction of $\ddot{\vec{p}}(t_0)$

$\rightarrow \vec{E}(r,\theta,t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}$; $\vec{B}(r,\theta,t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\phi}$

$\rightarrow P_{rad}(t_0) \cong \frac{\mu_0}{6\pi c}\left(\ddot{p}(t_0)\right)^2$

Radiation of point charge: $\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3} \left[\vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right]$

$$\vec{S}_{rad} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{\mathscr{r}}$$

Larmor formula: $P \approx \frac{\mu_0 q^2 a^2}{6\pi c}$ for $v \ll c$

Liénard's generalization: $P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$ for any speed v

$\frac{dW}{dt_r} = \left(\frac{\vec{\mathscr{r}}\cdot\vec{u}}{\mathscr{r}c}\right)\frac{dW}{dt}$ wit $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Power radiated into a patch of area $\mathscr{r}^2 \sin\theta \, d\theta \, d\phi = \mathscr{r}^2 d\Omega$ is:

$$\frac{dP}{d\Omega} = \left(\frac{\vec{\mathscr{r}}\cdot\vec{u}}{\mathscr{r}c}\right)\frac{1}{\mu_0 c} E_{rad}^2 \mathscr{r}^2 = \frac{q^2}{16\pi^2\varepsilon_0} \frac{|\hat{\mathscr{r}} \times (\vec{u}\times\vec{a})|^2}{(\hat{\mathscr{r}}\cdot\vec{u})^5}$$

Radiation reaction:

Abraham-Lorentz formula: $\vec{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}}$ (time average of parallel component)

↳ recoil force from radiation

→ The radiation reaction is due to the force of the charge on itself

H12 Electrodynamics and Relativity

Einstein: 1. The principle of relativity: the laws of physics apply in all inertial reference systems.

2. The universal speed of light: the speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source

$$\rightarrow V_{AC} = \frac{V_{AB} + V_{BC}}{1 + \frac{V_{AB} \cdot V_{BC}}{c^2}}$$

Time dilation: $\Delta \bar{t} = \sqrt{1 - \frac{v^2}{c^2}}\,\Delta t = \frac{\Delta t}{\gamma}$

Lorentz contraction: $\Delta \bar{x} = \underbrace{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}\Delta x = \gamma \Delta x$

for the observer on the moving object

Lorentz transformations: S stationary an $\bar{S}$ moving at velocity $v\hat{x}$

$$\Rightarrow \begin{bmatrix} \bar{x} = \gamma(x - v t) \\ \bar{y} = y \\ \bar{z} = z \\ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \end{bmatrix} \xrightarrow{\text{otherwise}} \begin{bmatrix} x = \gamma(\bar{x} + v\bar{t}) \\ y = \bar{y} \\ z = \bar{z} \\ t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \end{bmatrix}$$

$\bar{S}$ stationary and S moving like $-v\hat{x}$

Structure of Spacetime:

take $x^0 = ct$, $\beta = \frac{v}{c}$, $x^1 = x$, $x^2 = y$, $x^3 = z$ $\xrightarrow{\text{L-transformation}}$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

or in short $\bar{x}^\mu = \sum_{\nu=0}^{3} \underbrace{\left(\Lambda^\mu_\nu\right)}_{\text{Lorentz transformation matrix}} x^\nu$

Λ^μ_ν: $\mu \rightarrow$ row, $\nu \rightarrow$ column

$x^\mu = (\Lambda^{-1})^\mu_\nu \bar{x}^\nu$ ($\sum_\nu$)

Define 4-vector as any vector that transforms in the same way as (x^0, x^1, x^2, x^3) under Lorentz transformations

$$\rightarrow \bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu a^\nu$$

4-dimensional scalar product $= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$

covariant vector $a_\mu \neq$ contravariant vector a^μ

$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$ $\rightarrow a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu$ with $g_{\mu\nu} = \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Minkowski metric}}$

$\Rightarrow$ scalar product $= \sum_{\mu=0}^{3} a^\mu b_\mu = a^\mu b_\mu$

Invariant interval: $a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$

If $a^\mu a_\mu > 0 \rightarrow a^\mu$ is called spacelike

If $a^\mu a_\mu < 0 \rightarrow a^\mu$ is called timelike

If $a^\mu a_\mu = 0 \rightarrow a^\mu$ is called lightlike

Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$ and event B $(x_B^0, x_B^1, x_B^2, x_B^3)$

$\rightarrow$ displacement 4-vector $= \Delta x^\mu \equiv x_A^\mu - x_B^\mu$

$\rightarrow$ invariant interval $I \equiv (\Delta x)^\mu (\Delta x)_\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2$

with $t = t_A - t_B$ and $d = |\vec{r}_A - \vec{r}_B|$

$\rightarrow$ Lorentz transformation: $\bar{t} \neq t, \bar{d} \neq d$ BUT $\bar{I} = I$

Proper time and proper velocity:

your watch: $d\tau$
the clock on the wall: dt $\Big\}$ $d\tau = \sqrt{1 - \frac{u^2}{c^2}}\, dt$ with u your speed

ordinary velocity: $\vec{u} = \frac{d\vec{l}}{dt}$
proper velocity: $\vec{\eta} = \frac{d\vec{l}}{d\tau}$ $\Big\}$ $\Rightarrow \vec{\eta} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \vec{u}$

$\Rightarrow \eta^\mu = \frac{dx^\mu}{d\tau}$ (proper velocity 4-vector)

and $\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$ $\qquad \eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}}$

Energy and momentum: $\vec{p} = m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$; $p^\mu = m\eta^\mu \rightarrow p^0 = m\eta^0 = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}}$

relativistic energy $E = p^0 c = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$

$E_{rest} = mc^2$

$E_{kin} = E - mc^2 = mc^2 \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right)$

In every closed system, the total relativistic energy and momentum are conserved !

invariant quantity: same value in all inertial systems

$\neq$

Conserved quantity: same value before and after some process

$p^\mu p_\mu = -(p^0)^2 + (\vec{p} \cdot \vec{p}) = -m^2 c^2$

$\Rightarrow E^2 - p^2 c^2 = m^2 c^4$

Relativistic dynamics: $\vec{F} = \frac{d\vec{p}}{dt}$ relativistic momentum ; $W = \int \vec{F} \cdot d\vec{l}$

→ First and second of Newton's laws hold in special relativity

THE 3rd DOESN'T

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{F_y}{\gamma\left(1 - \beta \frac{u_x}{c}\right)} \; ; \; \bar{F}_z = \frac{d\bar{p}_z}{d\bar{t}} = \frac{F_z}{\gamma\left(1 - \beta \frac{u_x}{c}\right)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1 - \beta \frac{u_x}{c}} = \frac{F_x - \frac{\beta}{c}(\vec{F} \cdot \vec{u})}{1 - \beta \frac{u_x}{c}}$$

If a particle is at rest in S so that $\vec{u} = 0$

then $\vec{\bar{F}}_\perp = \frac{1}{\gamma} \vec{F}_\perp$

$\bar{F}_\parallel = F_\parallel$

↳ parallel to the moving direction of $\bar{S}$

Minkowski force: $K^\mu \equiv \frac{dp^\mu}{d\tau}$; $K^0 = \frac{dp^0}{d\tau} = \frac{1}{c} \frac{dE}{d\tau}$

$$\vec{K} = \left(\frac{dt}{d\tau}\right) \frac{d\vec{p}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}} \vec{F}$$

Total momentum: classic: $\vec{P} = M \frac{d\vec{R}_m}{dt}$ with center-of-mass $\vec{R}_m = \frac{1}{M} \sum m_i \vec{r}_i$

$\Updownarrow$

relativistic: $\vec{P} = \frac{E}{c^2} \frac{d\vec{R}_e}{dt}$ with center-of-energy $\vec{R}_e = \frac{1}{E} \sum E_i \vec{r}_i$

Field transformation: $\bar{S}$ moving to the right with respect to S with velocity $v\hat{x}$

$$\Rightarrow \quad \bar{E}_x = E_x \; ; \; \bar{E}_y = \gamma(E_y - vB_z) \; ; \; \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \; ; \; \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right) \; ; \; \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

Special cases: 1. If $\vec{B} = \vec{0}$ in S, then $\vec{\bar{B}} = -\frac{1}{c^2}(\vec{v} \times \vec{\bar{E}})$

2. If $\vec{E} = \vec{0}$ in S, then $\vec{\bar{E}} = \vec{v} \times \vec{\bar{B}}$

Field tensor:

$$\bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma} \; , \; t^{\mu\nu} = \begin{pmatrix} t^{00} & \cdots & t^{03} \\ \vdots & & \vdots \\ t^{30} & \cdots & t^{33} \end{pmatrix}$$

Symmetric tensor: $t^{\mu\nu} = t^{\nu\mu}$

antisymmetric ": $t^{\mu\nu} = -t^{\nu\mu}$

→ (antisymmetric) field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$F^{\mu\nu} \Longrightarrow$ substitution of $\vec{E}/c \to \vec{B}$, $\vec{B} \to -\vec{E}/c$ $\Longrightarrow$ dual tensor $G^{\mu\nu}$

$\rightarrow$ (antisymmetric) dual tensor $G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$

Both tensors generate the correct transformation rules for $\vec{E}$ and $\vec{B}$

Electrodynamics in tensor notation:

$$\rho = \frac{Q}{V} \; ; \; \vec{J} = \rho\vec{u} \; ; \text{ proper charge density } \rho_0 = \frac{Q}{V_0} \; ; \; V = \sqrt{1-\frac{u^2}{c^2}}\, V_0$$

$$\rightarrow \rho = \rho_0 \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \; ; \; \vec{J} = \rho_0 \frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow J^\mu = \rho_0 \eta^\mu = (c\rho, J_x, J_y, J_z)$$

$$\sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = 0 \; ; \; \sum_{\nu=0}^{3} \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \; ; \; \sum_{\nu=0}^{3} \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \; ; \; K^\mu = q \sum_{\nu=0}^{3} \eta_\nu F^{\mu\nu}$$

Remember:

$$\eta_\nu = (\eta_0, \eta_1, \eta_2, \eta_3) = (-\eta^0, \eta^1, \eta^2, \eta^3)$$

$$\rightarrow \vec{K} = \frac{q}{\sqrt{1-\frac{u^2}{c^2}}} \left(\vec{E} + (\vec{u} \times \vec{B}) \right)$$

Relativistic potentials:

$$A^\mu = \Big(\underbrace{V}_{\text{potential}}/c, A_x, A_y, A_z \Big)$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

x_ν $\rightarrow$ covariant vector !

Lorenz gauge condition: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \Rightarrow \sum_{\mu=0}^{3} \frac{\partial A^\mu}{\partial x^\mu} = 0$

$$\Rightarrow \Box^2 A^\mu = -\mu_0 J^\mu \text{ with } \Box^2 = \sum_{\nu=0}^{3} \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

1.6 (pg 52) Theory of vector fields

$\vec{\nabla} \times \vec{F} = \vec{0} \iff \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l}$ independent of path $\iff \oint \vec{F} \cdot d\vec{l} = 0 \iff \vec{F} = -\nabla V$

$\vec{\nabla} \cdot \vec{F} = 0 \iff \int \vec{F} \cdot d\vec{a}$ independent of surface $\iff \oint \vec{F} \cdot d\vec{a} = 0 \iff \vec{F} = \vec{\nabla} \times \vec{A}$

Binomial expansion: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Taylor expansion: $f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \left(\frac{d}{dx}\right)^n f \Big|_{x=a}$