Exam Stochastic Models
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K.U.Leuven

There are four questions. Answer each question on a separate page. This is a written exam, so try to provide your solutions with sufficient explanation. Write your name on each page; also on the question sheets, which should be handed in together with your solutions. This is an open book exam: you can use a pocket calculator, the syllabus, the handouts of the PowerPoint presentations, and any written solutions to the exercises discussed during the exercise sessions. You have 4 hours for the exam questions. If anything is not clear, please ask.

God speed!

1. Consider the Poisson process $N = \{N(t); t \geq 0\}$ with intensity λ.
   For $0 < s < t < u$ and $k < l < m$, determine

   (a) $\mathrm{E}\left[2^{N(t)} e^{-\lambda t} \,|N(s) = k\right]$;

   (Hint: $\mathrm{E}\left[e^{xN(t)}\right] = e^{-\lambda t(1-e^x)}$)

   (b) $\mathrm{Var}\left[N(t) \,|N(s) = k\right]$;

   (c) $\mathrm{Cov}\left[N(s), N(t)\right]$.

   (Hint: for any two r.v. X and Y it holds that $\mathrm{Cov}\left[X, Y\right] = \mathrm{E}\left[X \cdot Y\right] - \mathrm{E}\left[X\right] \cdot \mathrm{E}\left[Y\right]$)

   Prove that

   (d) $\Pr\left[N(t) = l \,|N(s) = k, N(u) = m\right] = \Pr\left[N(t-s) = l-k \,|N(u-s) = m-k\right]$.

2. A radio-show gives a present to every 10^{th} caller. Calls arrive according to a Poisson process with an average of λ calls per minute.

   (a) Let k be a nonnegative integer and $0 < s < t < u$. What is the expected number of calls that arrive in the time interval $(s, u]$ minutes, if you know that exactly k calls arrive during the first t minutes?

   (b) Suppose the 8^{th} call arrives after exactly 6 minutes. What is de probability that you should wait at least 9 minutes (in total) until the first present is given. Assume that on average there are 0.5 calls per minute.

   (c) Same question as in (b), but assume now that call-times satisfy the recurrence relation $T_{k+1} = (1 + \alpha_{k+1})\, T_k$, $k \geq 1$, where the a_k are i.i.d. Uniform$\left[0, \frac{1}{2}\right]$ r.v. and independent of the T_k.
   (Note that the inter-arrival times are no longer i.i.d. Exponential(λ), hence this is no longer a Poisson process.)

3. At the chicken-farm *Royal Chick* food supplies are stored in a storage room that can contain at most $2b - 1$ kilograms ($b = 1, 2, ...$). Every morning farmer Filip needs B kg of food for his chickens, with $B \in \{0, 1, ..., b\}$. The probability that Filip needs i kg of food equals p_i. Moreover, the amounts of food required at different days are mutually independent and independent of the amount of food available in the storage room. Furthermore, at the end of each day his wife Mathilde checks the storage room and refills it with exactly b kg of food, if the amount of food is strictly below b kg.

   Consider the DTMC $X = \{X_n; n = 0, 1, ...\}$, where X_n is the amount of food in the storage room at the beginning of day n, i.e. before Filip has fed his chickens.

   (a) Give the one-step transition matrix of X.

   (b) Suppose that Filip found this morning 5 kg of chicken food in the storage room. What is the expected number of days until he will find 3 kg of chicken food in the storage room?

   (c) What is the long-run fraction of the days that the storage room contains at least 2 kg, after Filip has fed his chickens?

4. A system consist out of N machines, of which in total $M \leq N$ machines can be operative and the rest remains stand-by. If a machine is operative, it keeps working for an arbitrary long time until it breaks down. Assume the time until a break-down is exponentially distributed with parameter μ. When a machine has broken down, it will be repaired. In total, R machines can be repaired simultaneously. The repair-time is exponentially distributed with parameter λ. Hence, a machine is either (1) working, (2) stand by, (3) in repair, (4) waiting for being repaired. Moreover, one always works at maximum capacity, i.e. have as many machines operative as possible ($\leq M$) and repair as many machines simultaneously as possible ($\leq R$).

   Consider the CTMC $X = \{X_t; t \geq 0\}$, where X_t is the number of machines that are operative or stand-by.

   (a) Give the infinitesimal generator of X.

   (b) Take $M = N = R$. With the passage of time, determine the probability that none of the machines is still working.

   (c) Take again $M = N = R$. What is the expected length of a period during which at least one machine is operative?