Take-home assignment Functional Analysis

Solutions to be handed in before November 17, 2014, 12:00, to me personally, in my mailbox (on the ground floor), or by sending them as a PDF-file to `tim.delaat@wis.kuleuven.be` .

Note: This take-home assignment counts for 8 out of 20 points on the January 2015 exam.

Rules for collaboration. You are encouraged to work together, but you have to hand in your own solutions. Your answers have to be formulated in your own words. Note that it is highly unlikely that, even in a close collaboration, two students come up with exactly the same computations and notation.

Literature. You are encouraged to consult books on functional analysis, e.g., the ones by Conway and Pedersen. Please make sure you refer correctly when using a result from the literature.

Here, all Banach spaces are defined over $\mathbb{C}$ and the natural numbers $\mathbb{N}$ are assumed to include 0.

1 Generalizing trace-class and Hilbert-Schmidt operators

Study yourself Section 3.6 of the lecture notes.

Definition 1.1. Let H be a separable Hilbert space. For $p = 1, 2, 3, \ldots$, let

$$S^p(H) = \{T \in \mathcal{K}(H) \mid |T|^p \text{ is trace-class }\}.$$

Note: The spaces $S^p(H)$ can be defined for all $p \geq 1$. For this, however, we need another version of the functional calculus than the one that was proved in the lectures.
By Theorem 3.22 in the lecture notes, the space $\mathcal{TC}(H)$ of trace-class operators on H corresponds to $S^1(H)$, and the space $\mathcal{HS}(H)$ of Hilbert-Schmidt operators on H corresponds to $S^2(H)$.

- **(i)** Show that with the norm given by $\|T\|_p = (\mathrm{Tr}(|T|^p))^{\frac{1}{p}}$, the space $S^p(H)$ is a Banach space for all $p = 1, 2, 3, \ldots$.
  Hint: Do not prove this directly from the definition of Banach space. Indeed, showing the triangle inequality for the above norm is not easy.

Let $H = \ell^2(\mathbb{N})$, let $\varphi \in \ell^\infty(\mathbb{N})$, and let $M_\varphi \colon H \to H$ be the multiplication operator associated with φ (see Example 2.2 of the lecture notes). We have seen that M_φ is a bounded operator.

- **(ii)** Show that M_φ is a compact operator if and only if $\varphi \in c_0(\mathbb{N})$. (See Definition 0.8 in the lecture notes for the definition of $c_0(\mathbb{N})$.)

- **(iii)** For $p = 1, 2, 3, \ldots$, show that $M_\varphi \in S^p(\ell^2(\mathbb{N}))$ if and only if $\varphi \in \ell^p(\mathbb{N})$.

- **(iv)** It follows from the previous part of this exercise that for $p = 1$, the operator M_φ is trace-class. Compute its trace in this case.

In the same way as for the trace-class operators and the Hilbert-Schmidt operators, it can be shown that $S^p(H)$ is a two-sided ideal of $B(H)$. An important ingredient in the proof is the polar decomposition (see Theorem 3.21 of the lecture notes).

- **(v)** For $\varphi \in \ell^\infty(\mathbb{Z})$, consider the operator $T \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by $(Tx)(n) = \varphi(n)x(n+1)$. Show that T is a bounded operator and compute its polar decomposition.

2 Complemented subspaces and projections on Banach spaces

You are free to consult books, e.g., the ones by Conway and Pedersen. However, note that they may use slightly different conventions. In your solutions to this assignment, make sure you use the conventions used below.

Definition 2.1. Let X be a Banach space. A closed linear subspace V of X is said to be complemented in X if there exists a closed linear subspace W of X such that $X = V + W$ and $V \cap W = \{0\}$. We write $X = V \oplus W$.

Let $c(\mathbb{N})$ denote the space of functions $x\colon \mathbb{N} \to \mathbb{C}$ such that $\lim_{n\to\infty} x(n)$ exists. When equipped with the norm $\|.\|_\infty$, the space $c(\mathbb{N})$ becomes a Banach space.

(i) Show directly (from the definition above) that $c_0(\mathbb{N})$ is complemented in $c(\mathbb{N})$.

It is known that not all closed linear subspaces in a Banach space are complemented. For example, it is well-known (but difficult to prove!) that $c_0(\mathbb{N})$ is not complemented in $\ell^\infty(\mathbb{N})$.

It is clear that if V is complemented in X by W, then for every $x \in X$, there exist unique $v \in V$ and $w \in W$ such that $x = v + w$.

So far, our notion of complemented subspace is purely algebraic. However, we can link it to continuous projections on Banach spaces.

Definition 2.2. A projection on a vector space X is a map $E\colon X \to X$ such that $E^2 = E$.

Note that this generalizes the notion of orthogonal projection on a Hilbert space.

(ii) Show that if $E\colon X \to X$ is a continuous linear projection on a Banach space X, then $\mathrm{Ker}(E)$ and $\mathrm{Im}(E)$ are closed linear subspaces of X.

(iii) For $p \in [1,\infty]$, show that $p\colon f \mapsto \chi_{[0,1]} f$ defines a continuous linear projection on $L^p(\mathbb{R})$. Compute the norm of p and determine its image.

We can now turn to the relation between complemented subspaces and projections on Banach spaces.

(iv) Let V be a closed linear subspace of a Banach space X. Prove that V is complemented in X if and only if there exists a bounded projection $E\colon X \to X$ with image V. Show that in this case, V is in fact complemented by $\mathrm{Ker}(E)$.

3 Around the great Banach space theorems

(i) Let X and Y be normed spaces, and let $T\colon X \to Y$ be a linear operator. Show that the graph of T is closed if and only if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \to 0$ as $n \to \infty$ and $Tx_n \to y$ for some $y \in Y$, then $y = 0$.

(ii) Let X be a vector space with two norms $\|.\|_1$ and $\|.\|_2$ with corresponding topologies τ_1 and τ_2, respectively, such that X is complete with respect to both these norms and such that there exists an $M > 0$ such that $\|x\|_1 \le M\|x\|_2$ for all $x \in X$. Show that $\tau_1 = \tau_2$, i.e., $\|.\|_1$ and $\|.\|_2$ induce the same topologies.
Hint: Prove and use the fact that two norms $\|.\|_1$ and $\|.\|_2$ induce the same topology if and only if there exist constants $\alpha, \beta > 0$ such that $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$ for all $x \in X$. Such norms are said to be equivalent.

(iii) A function $f\colon \mathbb{C} \to X$ is said to be weakly holomorphic if for all $\varphi \in X^*$, the function $\varphi \circ f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic. Prove the following generalization of Liouville's theorem: if $f\colon \mathbb{C} \to X$ is a weakly holomorphic function that is bounded, i.e., there exists an $M > 0$ such that $\|f(z)\| \le M$ for all $z \in \mathbb{C}$, then f is constant.

(iv) Let $1 \leq p \leq \infty$, and let (A_{ij}) be a matrix (with $i, j \in \mathbb{N}$) such that $(A\varphi)(i) = \sum_{j=1}^{\infty} A_{ij}\varphi(j)$ defines an element $A\varphi$ of $\ell^p(\mathbb{N})$ for every φ in $\ell^p(\mathbb{N})$. Show that $A \in B(\ell^p(\mathbb{N}))$.