Symmetries in Quantum Mechanics

(14/01/2013 (13u30-20u45))

1 A carbon dioxide molecule can be modeled as three interacting point particles (two of mass m_{O}, one of mass m_{C}), with postion vectors $\vec{r}_{\mathrm{O}_1}$, $\vec{r}_{\mathrm{C}}$, $\vec{r}_{\mathrm{O}_2}$, which in static equilibrium are lying on a line, arranged as $\mathrm{O}-\mathrm{C}-\mathrm{O}$. In (x,y,z) coordinates such an equilibrium configuration is given e.g. by $\vec{R}_{\mathrm{O}_1}=(0,0,-a)$, $\vec{r}_{\mathrm{C}}=(0,0,0)$, $\vec{r}_{\mathrm{O}_2}=(0,0,+a)$. The nontrivial vibrational normal modes of the molecule around this static equilibrium configuration are given by two transversal modes T_x and T_y, a symmetric longitudinal mode L_s and an antisymmetric longitudinal mode L_a.

(a) What is the spatial symmetry group G of the equilibrium configuration?

(b) Write out the four normal modes explicitly, i.e. as $\vec{r}_{\mathrm{C}}=T_x\left(t\right)\vec{v}_{x,\mathrm{C}}+T_y\left(t\right)\vec{v}_{y,\mathrm{C}}+L_s\left(t\right)\vec{v}_{s,\mathrm{C}}+L_a\left(t\right)\vec{v}_{a,\mathrm{C}}$ etc for suitable constant vectors $\vec{v}_{x,\mathrm{C}}$, $\vec{v}_{x,0_1}$, etc, with $T_x\left(t\right)=\sin\left(\omega_x t+c_x\right)$ etc. No need to relate the mode frequencies to morge fundamental quantities, but do show that these modes are indeed decoupled.

(c) Discuss how the normal modes transform under the symmetry group G of the equilibrium configuration.

(d) Find the quantum normal mode energy eigenfunction $\psi\left(T_x,T_y,L_s,L_a\right)$, which will be characterized by four quantum numbers (n_1,n_2,n_3,n_4), and discuss how the symmetry group G is represented on the energy levels.

(e) Discuss the selection rules for matrix elements of the form

$$\langle n_1,n_2,n_3,n_4|\vec{E}\cdot\left(\vec{r}_{\mathrm{O}_1}+\vec{r}_{\mathrm{C}}+\vec{r}_{\mathrm{O}_2}\right)|n_1',n_2',n_3',n_4'\rangle.$$

Here $\vec{E}$ is some constant vector. (Such matrix elements appear e.g. in radiative dipole transition rates.)

2 Come up with an original (not just copied from the book) exam-type problem relevant to the material covered in this course and solve it.