Calculus I, **28/11/2016**

Reeksnr.:	
Naam:	*XXXXXXXXXXXXXXX*

1. Beschouw de functie

$$F(x) = -\int_{-x^2}^{0} e^{\tan^{-1} t}\, dt$$

   (a) Benoem het domein van deze functie, bereken de kritieke punten, en geef aan waar eventuele globale/lokale extrema zich bevinden. Bespreek verder ook het concaaf/convex karakter van de grafiek van $F(x)$, en geef aan of er al dan niet buigpunten zijn. [opmerking: je hoeft de integraal zelf niet uit te rekenen!]

   (b) Argumenteer vanuit het (asymptotisch) gedrag van de integrand ($e^{\tan^{-1} t}$) dat deze functie voldoet aan

$$\lim_{x\to-\infty} F(x) = -\infty$$

   Wat besluit je dan voor $\lim_{x\to+\infty} F(x)$?

(2 ptn)

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x

Antwoord:

(a) $DOM \equiv \Re$ (all the involved functions have domain $\Re$). **Typical error: the domain of the arctangent is NOT $[-\pi/1, \pi/2]$. This is the range of the function. Also, then notation $\tan^{-1} x$ indicates the arctangent of x, not the cotangent; for that, the notation $\cot x$ is used.**

Differentiate: $F'(x) = -\left[e^{\tan^{-1} t}\Big|_{t=0} \frac{d}{dx}(0) - e^{\tan^{-1} t}\Big|_{t=-x^2} \frac{d}{dx}(-x^2) \right] = -2xe^{\tan^{-1}(-x^2)}$. **Typical errors: the derivative of an integral function is the function itself evaluated at the integral range limits times the derivative of the limits (Leibniz rula as a consequence of the chain differentiation rule). Watch out for the minus sign! If left out, critical points become the opposite of what they should be.**
$F'(x) = 0$ only if $x = 0$, and $F'(x)$ has the same sign of $-x$ $\forall x \in \Re$. Therefore, $F(x)$ increases for $x < 0$ and decreases for $x > 0$. We conclude that $x = 0$ is a point of global maximum. **Typical error: characterization of the critical point must be explicitly stated. If not, a maximum could be local but not global.**

Differentiate again to find $F''(x) = -2e^{\tan^{-1}(-x^2)} \frac{(x^2-1)^2}{x^4+1}$.
$F''(x) = 0$ only if $x = \pm 1$, and $F''(x) < 0$ for any other value of x. It follows that $F(x)$ is concave everywhere with no inflection points. **Typical error: the fact that the second derivative vanishes for some value of x DOES NOT imply the presence of inflection points. Inflection points are present if and only if the second derivative changes sign. Here, the second derivative vanishes at $x = \pm 1$, but it does not change sign around these points, therefore there are no inflection points in $F(x)$. Also, a function is conventionally concave if its concavity points downwards (thus $F''(x) < 0$), and vice versa; not the other way around.**

(b) We know that

$$\lim_{t\to-\infty} e^{\tan^{-1} t} = e^{-\pi/2},$$

and since $e^{\tan^{-1} t}$ is monotonically increasing in $\Re$, it follows that $e^{\tan^{-1} t} \geq e^{-\pi/2}$ $\forall t \in \Re$. **Typical error: the limit for the arctangent for x going to $\pm\infty$ is NOT $\pm\infty$, but rather $\pm\pi/2$.** This implies that $\int e^{\tan^{-1} t} dt \geq \int e^{-\pi/2} dt$ (this can be inferred by comparing the areas of the

plane region under the lines $y = e^{\tan^{-1} x}$ and $y = e^{-\pi/2}$, respectively).
Since we know

$$\lim_{x \to \pm\infty} \int_{-x^2}^{0} e^{-\pi/2} dt = \infty,$$

it follows immediately that

$$\lim_{x \to \pm\infty} \int_{-x^2}^{0} e^{\tan^{-1} t} dt = \infty.$$

Therefore,

$$\lim_{x \to \pm\infty} F(x) = -\infty.$$

2. De som

$$S_n = \sum_{i=1}^{n} \frac{4i}{n^2} \ln\left(1 + \frac{2i}{n}\right)$$

is een Riemannsom voor een functie $f(x)$ op het interval $[0, 2]$.

   (a) Schrijf $\lim_{n\to\infty} S_n$ als een bepaalde integraal op het interval $[0, 2]$

   (b) Reken de integraal uit die je in (a) vond.
   Als je opgave (a) niet kon oplossen, bereken dan $\int_0^1 \ln(x^2 + 1)\, dx$. (**Opmerking:** deze integraal is niet het antwoord op vraag (a))

(1.5 ptn)

3. Bereken het oppervlak van het omwentelingslichaam bekomen door het stuk van kromme $\mathcal{C}$ gegeven via $9x^2 = 4y^3$ tussen $(0, 0)$ en $(\frac{2}{3}, 1)$ te roteren rond de x-as. *(1.5 ptn)*

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x

Antwoord voor vragen 2 en 3:

2. (a) We can rewrite the sum as

$$S_n = \sum_{i=1}^{n} \frac{2}{n}\frac{2i}{n} \ln\left(1 + \frac{2i}{n}\right).$$

Now assigning $x = 2i/n$, it is clear that, for $n \to \infty$, the sum represents a discrete form of the integral

$$\int_0^2 x \ln(1 + x) dx.$$

(b) We can integrate e.g. by parts by assigning $u = \ln(1 + x)$ and $dv = x dx$, so that

$$\int_0^2 x \ln(1 + x) dx = \frac{x^2}{2} \ln(1 + x)\bigg|_0^2 - \int_0^2 \frac{x^2}{2} \frac{dx}{1 + x}.$$

The integrand in the second term can be reduced to a simpler form by division of polynomials, obtaining

$$\frac{x^2}{2} \ln(1 + x)\bigg|_0^2 - \frac{1}{2} \int_0^2 \left(x - 1 + \frac{1}{1 + x}\right) dx.$$

Finally, we get

$$\frac{x^2}{2} \ln(1 + x)\bigg|_0^2 - \frac{1}{2} \left(\frac{x^2}{2} - x + \ln(1 + x)\right)\bigg|_0^2 = \frac{3}{2} \ln 3.$$

Typical error: $\ln(1) = 0$.

The solution of the alternative integral can be carried out e.g. by parts, by assigning $u = \ln(1+x^2)$ and $dv = dx$. Therefore,

$$\int_0^1 \ln(1 + x^2) dx = x \ln(1 + x^2)\big|_0^1 - \int_0^1 \frac{2x^2}{1 + x^2} dx.$$

Again, the integrand can be simplified by division of polynomials and then easily solved, so that

$$x \ln(1 + x^2)\big|_0^1 - 2 \int_0^1 \left(1 - \frac{1}{1 + x^2}\right) dx = x \ln(1 + x^2)\big|_0^1 - 2 \left(x - \tan^{-1} x\right)\big|_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

3. Rewriting the equation of the curve in terms of $x = g(y)$ (writing it as $y = f(x)$ gives out an integral which is much more difficult to solve):

$$x = \pm\frac{2}{3}\sqrt{y^3} \Rightarrow \frac{dx}{dy} = \pm\sqrt{y}.$$

The line element is given by $ds = \sqrt{1 + (dx/dy)^2}dy = \sqrt{1+y}dy$. We use this to calculate the area of the surface of the solid of revolution obtained by rotating the given curve about the x-axis with the usual formula,

$$S = 2\pi \int_{y=0}^{y=1} |y| ds = 2\pi \int_0^1 y\sqrt{1+y}dy.$$

This can be integrated, e.g. by parts, to obtain

$$S = 2\pi \left[y\,\frac{2}{3}(1+y)^{3/2}\Big|_0^1 - \frac{2}{3}\int_0^1 (1+y)^{3/2}dy \right] = 2\pi \left[\frac{2}{15}(1+y)^{3/2}(3y-2) \right]\Big|_0^1 = \frac{8\pi}{15}(\sqrt{2}+1).$$

Typical error: it is requested to calculate the AREA of the surface of revolution, not its enclosed VOLUME. The formulas are different.