Examen General Relativity: 30 januari 2018

Question 1: Wormhole

(a) Discribe the main difference between a free motion of a particle in Newtonian mechanics and in General Relativity.

(b) *This was an example from the book.* Given is the metric of a wormhole.

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

Determine the geodisic equations and calculate the proper time of an observer falling radially through the wormhole staring at a position $r = R$ with velocity U to position $r = -R$.

Question 2: Generalised Schwarzschild metric

Given is the generalised Schwarzschild metric, with Λ the cosmological constant.

$$ds^2 = -\Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{2}$$

(a) Determine an expression for the radial component $r(\lambda)$ for a particle with λ an affine parameter in terms of the effective potential ($V_{\text{eff}}(r)$).

(b) For Λ positive or negative, what are the modifications for the bound orbits?

(c) Take $\Lambda = 0$. Describe what happens when ε is exactly equal to the maximum of $V_{\text{eff}}(r)$. What happens to the orbit when ε is just a little higher or smaller than $V_{\text{eff}}(r)$.

(d) *This is an exercise in the book.* Take $\Lambda = 0$. What is the longest proper time that an observer can spend, just passing the horizon, before he gets destroyed in the singularity?

(e) *Also an exercise in the book, I think.* Again take $\Lambda = 0$. When an observer falls radially inwards a black hole, starting with zero kinetic energy at infinity, what is the time that passes on the observers clock between position $r = 6M$ and $r = 2M$.

Question 3: Cosmology

Given is the line element a Robertson-Walker (RW) universe

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2\Big) \tag{3}$$

and the Friedmann-Lemaître equation

$$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - k. \tag{4}$$

(a) Rewrite (4) in terms of $U_{\text{eff}}(a)$ for the scale factor and show for a given Λ there is a critical value of ρ_m for which Λ doen not change in time. Find this. What is the spatial volume of the universe in terms of Λ?

(b) Describe with a causal diagram what horizon means in cosmology. Derive an equation for the physical distance $d_{\text{hor}}(t)$ for flat space when matter is dominating. Determine $d_{\text{hor}}(t_0)$ and the age of the universe for t_0 and write your results in terms of the Hubble constant H_0, which is (*value was given*). If $t_0 \sim 9$ Gyr, how come we find galaxies that are older than this t_0?

(c) If $\rho + 3p > 0$ for a flat space RW universe, does this model predict a big bang singularity in the beginning of time? Is this the case for our universe?