$\boxed{1}$ Random walks, diffusion and polymers

statistical mechanics: bridge between microscopic and macroscopic descriptions

set-up: fluid composed of a large number of identical particles
at any temp $T>0$: particles subject to thermal motion
we're interested in a test particle (larger than fluid part)
we want to describe the dynamics of this test particle at sufficiently long time scales so that an average large number of collisions with fluid particles has occurred
→ displacement test particle : distance **a**, randomly

Jump size distribution

random walk = path in space generated by series of steps each selected independently from a given probability distribution $p(\vec{r})$
↳ start walk @ origin @ $t=0$, then at times $\Delta t, 2\Delta t, 3\Delta t$ ... → random steps $\vec{r}_1, \vec{r}_2, \vec{r}_3$ ... selected from $p(\vec{r})$

probability distribution $p(\vec{r})$ ⇒ $p(\vec{r})dxdydz$ is the prob that $\vec{r}=(x,y,z)$ is selected from interval $[x, x+dx], [y, y+dy], [z, z+dz]$

properties: $p(\vec{r}) \geq 0$, $\int p(\vec{r})d\vec{r} = 1$ (normalization)
$\langle\vec{r}\rangle = \int p(\vec{r})\vec{r}\,d\vec{r} = 0$, $\langle\vec{r}^2\rangle = \int p(\vec{r})\vec{r}^2 d\vec{r} = a^2$
(unbiased) (finite value mean squared jump)
↳ $p(\vec{r})$ has to decay fast @ infinity

examples:
1) $p(x) = \frac{1}{2}[\delta(x-a) + \delta(x+a)]$ → RW in a 1D lattice
2) $p(x) = \frac{1}{\sqrt{2\pi a^2}} e^{-x^2/2a^2}$ → RW w Gaussian distr. steps
3) $p(x) = \begin{cases} \frac{1}{2\sqrt{3}a} & |x| \leq \sqrt{3}a \\ 0 & \text{else} \end{cases}$ → uniform distribution
4) $p(\vec{r}) = \frac{1}{4\pi a^2}\delta(|\vec{r}|-a)$ → freely jointed chain (FJC)

End-to-end vector

end-to-end vect. measures the distance of the walk from the origin
→ after N steps: $\langle\vec{R}_e\rangle = \langle\sum_{i=1}^N \vec{r}_i\rangle$
$= \sum_{i=1}^N \langle\vec{r}_i\rangle = \sum_{i=1}^N \int d\vec{r}_i\,\vec{r}_i\,p(\vec{r}_i) = 0$
average displacement is 0 by symmetry (unbiased prop.)
→ squared end-to-end distance:
$\langle\vec{R}_e^2\rangle = \langle\sum_{i,j=1}^N \vec{r}_i\cdot\vec{r}_j\rangle = \sum_{i=1}^N\langle\vec{r}_i^2\rangle + \sum_{i\neq j}\langle\vec{r}_i\cdot\vec{r}_j\rangle = \sum_{i=1}^N\langle\vec{r}_i^2\rangle = a^2N$
since jumps are selected independently and thus $\langle\vec{r}_i\cdot\vec{r}_j\rangle = \langle\vec{r}_i\rangle\cdot\langle\vec{r}_j\rangle = 0$ if $i\neq j$
↳ universal equation (indep. of spec. distr.)

Diffusion equation

end-to-end vector prob. distr.: $P(\vec{R},t)$ = prob that the walk which started @ origin @ $t=0$ arrives @ $\vec{R}$ @ time t , $t=N\Delta t$

→ $P(\vec{R},t)$ is a solution of $\frac{\partial P(\vec{R},t)}{\partial t} = D\nabla^2 P(\vec{R},t)$ the diffusion equation

$P(\vec{R}, t+\Delta t) = \int d\vec{r}\, P(\vec{R}-\vec{r}, t)\, p(\vec{r})$ [RW in a d-dim. space]

integrate over all possible values of $\vec{r}$

= prob of being in $\vec{R}-\vec{r}$ @ t multiplied by the prob. of making a jump of $\vec{r}$

→ prod. of prob since each new jump is indep from the previous ones

$$\approx \int d\vec{r} \left\{ P(\vec{R},t) - \vec{r}\cdot\nabla P(\vec{R},t) + \frac{1}{2}\sum_{k,l} r_k r_l \frac{\partial^2 P(\vec{R},t)}{\partial R_k \partial R_l} \right\} p(\vec{r})$$

at long times $P(\vec{R},t)$ varies slowly compared to typical jumpsize a

$P(\vec{R},t)$ a smooth function at long t

→ (expansion in $\vec{r}$ & considered lower order terms)

$\int d\vec{r}\, p(\vec{r}) = 1$

$\int d\vec{r}\, \vec{r}\, p(\vec{r}) = \langle \vec{r} \rangle = 0$

$$= P(\vec{R},t) + \frac{a^2}{2d}\nabla^2 P(\vec{R},t)$$

(used isotropy property for d-dim walks: $\langle r_k r_l \rangle = \delta_{kl}\frac{a^2}{d}$)

→ for small Δt:

$$\frac{\partial P(\vec{R},t)}{\partial t} = \frac{a^2}{2d\Delta t}\nabla^2 P(\vec{R},t)$$

→ diffusion eq. with $D = \frac{a^2}{2d\Delta t}$

→ for N part syst: $D = \frac{a^2 N}{2d\,\Delta t}$

↳ describes the behaviour of the collective motion of the fluid particles resulting from the random walk movements of each particle → diffusion of particles

- a RW describing the traj. of a part in time:
  → zig-zag motion = Brownian motion → the presumably random moving of particles suspended in a fluid resulting from their bombardment by the fast-moving particles (at/mol.) in the fluid
- the diffusion eq. describes the time evolution of the concentration of particles spreading on some medium, each indiv. part performing a Brownian motion
  → M non-interacting, Brownian part concentration $c(\vec{R},t) = M\, P(\vec{R},t)$
- rewrite diff. eq: $\frac{\partial c(\vec{R},t)}{\partial t} = -\vec{\nabla}\cdot\vec{j}(R,t)$ with current $\vec{j}(\vec{R},t) = -D\nabla c(\vec{R},t)$
  → continuity eq.: describes time evolution conserved quantity, here: $\int d\vec{R}\, P(\vec{R},t)$ is cts in time & conservation of particles
- reaction - diff equation
  $$\frac{\partial c(\vec{R},t)}{\partial t} = D\nabla^2 c(\vec{R},t) + \underbrace{f(c(\vec{R},t))}_{\text{describes chemical reactions}}$$
  general → total concentration not conserved
  → $f(c)$ a non-linear function
- general form diff eq
  $$\frac{\partial c(\vec{R},t)}{\partial t} = \vec{\nabla}\left\{ D(c(\vec{R},t))\,\vec{\nabla} c(\vec{R},t) \right\}$$
  → diff coeff D can depend on concentration and position

Solving the diffusion equation

- Gaussians

$$G_{\vec{R}_0}(\vec{R},t) = \left(\frac{1}{4\pi D t}\right)^{\overset{\text{dimension}}{d/2}} e^{-\frac{(\vec{R}-\vec{R}_0)^2}{4Dt}}$$

$P(\vec{R},t) \xrightarrow[\text{for } t \gg 1]{}$

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (\text{for 1D})$$

$\lim_{t\to 0} G_{\vec{R}_0}(\vec{R},t) = \delta(\vec{R}-\vec{R}_0)$ | start from localized δ at t=0 (peak)

and var = 2Dt $(\sigma = \sqrt{2Dt})$ | spread out later in time $\sim\sqrt{t}$

↳ linear combinations $G_{\vec{R}_0}$ also solutions since diff eq is linear

most general solution: $c(x,t) = \int dy\, c(y)\, G_y(x,t)$

with right BC: $\lim_{t\to 0} c(x,t) = c(x) = \int dy\, c(y)\, \delta(y-x)$

Drift-diffusion equation

- external force acting on diff part → additional current j_F

$$j_F = v_d(x)\,c(x,t) = \frac{1}{\gamma} c(x,t) F(x) = -\frac{1}{\gamma} c(x,t)\frac{dV}{dx}$$

with v_d = drift velocity, γ = friction coeff, V(x) the potential, since $F = \gamma v$

→ friction/viscous forces dominate neglect initial acceleration

→ drift-diff eq:

$$\frac{\partial c(x,t)}{\partial t} = -\frac{\partial j_{tot}}{\partial x} = D\frac{\partial^2 c(x,t)}{\partial x^2} + \frac{1}{\gamma}\frac{\partial}{\partial x}\left\{ c(x,t)\frac{dV}{dx} \right\}$$

$$D\frac{\partial}{\partial x}\left\{ e^{-\beta V(x)}\left(\beta e^{\beta V(x)}\frac{dV}{dx}c(x,t) + e^{\beta V(x)}\frac{\partial c(x,t)}{\partial x}\right)\right\}$$

$$= D\frac{\partial^2 c}{\partial x^2} + D\beta\frac{\partial}{\partial x}\left(c(x,t)\frac{dV}{dx}\right)$$

- in total equilibrium: net total current vanishes

$$j_{tot} = j_D + j_F = -D\frac{dc_{eq}(x)}{dx} - \frac{1}{\gamma}c_{eq}(x)\frac{dV}{dx} = 0$$

$-A\exp(-)\frac{1}{k_BT}\frac{dV}{dx}$

$A\exp(-)\frac{dV}{dx}$

with $c_{eq}(x) = A\exp\left(\frac{-V(x)}{k_BT}\right)$ ✱ form solution T temp, k_B Boltzmann, A norm.

↳ distr expected for non-interacting part in eq. @ temp T

$$\Rightarrow D = \frac{k_BT}{\gamma}$$

Einstein relation → drift-diff → $\frac{\partial c(x,t)}{\partial t} = D\frac{\partial}{\partial x}\left\{ e^{-\beta V(x)}\frac{\partial}{\partial x}\left(e^{\beta V(x)}c(x,t)\right)\right\}$

2 Ensembles in classical statistical mechanics

system of N particles for which position and momenta are given (microscopic)

Introduction

microscopic state of a syst. defined by a 6N-dim vector $\Gamma \equiv (\vec{p}_1, \vec{p}_2, \dots, \vec{p}_N, \vec{q}_1, \vec{q}_2, \dots, \vec{q}_N)$

→ Γ-space = 6N dim vector space

Hamiltonian $\mathcal{H}$ governs time-evolution

$$\mathcal{H}(\Gamma) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \phi(\vec{q}_1, \vec{q}_2, \dots, \vec{q}_N)$$

observables $A(\Gamma)$ (functions in Γ-space)

ex kin. energy $A(\Gamma) = \sum_{i=1}^{N} \vec{p}_i^{\,2}/2m$

tot. energy $A(\Gamma) = \mathcal{H}(\Gamma)$

part. density $A(\Gamma, \vec{r}) = \sum_{i=1}^{N} \delta(\vec{q}_i - \vec{r})$ part @ a given pos.; for a homogeneous syst $\langle \rho_{\vec{r}}(\vec{r}) \rangle$ a constant indep. on $\vec{r}$

→ average values

- pure $\langle A \rangle_{pure} \equiv \frac{1}{\tau} \int_{t_0}^{t_0+\tau} A(\Gamma(t))\, dt$ (time average; with Γ(t) a path in Γ-space, solution to eq. of motion)

  ~ experiments: → if in equil.: ⟨A⟩ indep on t_0 and τ

  t_0 = time @ which measurement started, τ = duration → ⟨A⟩ = average measured value observable

- SM $\langle A \rangle = \int d\Gamma\, \rho(\Gamma) A(\Gamma)$

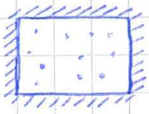
  ensemble average, using prob distr funct ρ(Γ) which represents the prob of finding a system in a microstate characterized by Γ

  3 different types of ρ(Γ) ⇝ microcanonical ensemble
  canonical ensemble
  grandcanonical ensemble

  in the limit of N → ∞ ⇒ all ρ(Γ) lead to the same value of thermal averages

  ↳ in principle we can use any of the 3 ensembles to calc thermal av., in practice canonical more convenient

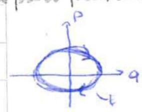
Microcanonical ensemble



isolated syst with fixed volume V, fixed # part N and NO energy exchange

→ total mechanical energy E conserved

→ part trajectories found in manifolds of constant tot. energy, ex (HO)



for a fixed energy, ρ_{mc} is constant along a part path in Γ

* PRINCIPLE OF EQUAL A PRIORI PROBABILITY (ESM, general)

All the microstates with a constant E are equally probable

→ prob. dens $\rho_{mc}(\Gamma) = \frac{\delta(E - \mathcal{H}(\Gamma))}{\omega(E,N,V)\, N!\, h^{3N}}$ N = # part, h = Planck's constant

$$= \underbrace{\frac{1}{N!\, h^{3N}}}_{} \underbrace{\frac{\delta(E - \mathcal{H}(\Gamma))}{\int \frac{d\Gamma}{N!\, h^{3N}} \delta(E - \mathcal{H}(\Gamma))}}_{\text{"norm."}}$$

QM: principle of indistinguishability of identical particles

QM

→ necessary for extensiveness thermodyn. quantities

$\omega(E) = \int \frac{d\Gamma}{N!\, h^{3N}} \delta(E - \mathcal{H}(\Gamma))$

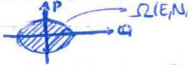
↳ microcanonical dens. of states

~ 6N-1 dim "volume" of manifold E = cte in Γ-space

→ $\Omega(E,N,V) = \int \frac{d\Gamma}{N!\, h^{3N}} \theta(E - \mathcal{H}(\Gamma))$

↳ step function $\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

~ volume of region $H(\Gamma) \leq E$



with $\omega(E,N,V) = \frac{\partial \Omega(E,N,V)}{\partial E}$

$\approx \frac{\Omega(E) - \Omega(E - \Delta E)}{\Delta E}$ for N >> 1, N ~ 10^{23} negligible

→ $\Omega(E,N,V) \approx \omega(E,N,V)\, \Delta E$

- connecting microscopic to macroscopic

BOLTZMANN entropy $S(E,N,V) = k_B \log \omega(E,N,V)$

$\approx k_B \log \Omega(E,N,V)$

example : Ideal gas

$$\mathcal{H}(\Gamma) = \Sigma_i \frac{\vec{p}_i^{\,2}}{2m} + \Sigma_i \varphi_{wall}(\vec{q}_i)$$

with $\varphi_{wall}(\vec{q}_i) = \begin{cases} 0 & \text{if } \vec{q}_i \in V \\ +\infty & \text{else} \end{cases}$ → confinement (fixed volume microcan.)

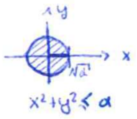
$$\rightarrow \Omega(E,N,V) = \int \frac{d\Gamma}{N!h^{3N}} \Theta\left(E - \Sigma_i \frac{\vec{p}_i^{\,2}}{2m} - \Sigma_i \varphi_{wall}(\vec{q}_i)\right)$$

$$= \frac{1}{N!h^{3N}} \int_V d\vec{q}_1 \int_V d\vec{q}_2 \cdots \int_V d\vec{q}_N \int d\vec{p}_1 \ldots d\vec{p}_N \, \Theta\left(E - \Sigma_i \frac{\vec{p}_i^{\,2}}{2m}\right)$$

$$= \frac{1}{N!h^{3N}} V^N \int d\vec{p}_1 \ldots d\vec{p}_N \, \Theta\left(E - \Sigma_i \frac{\vec{p}_i^{\,2}}{2m}\right)$$

$$= \frac{V^N}{N!h^{3N}} \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} (2mE)^{3N/2}$$

$\int dx dy \, \Theta(\alpha - (x^2+y^2)) = \pi\alpha$



$x^2+y^2 \leq \alpha$

for d-dim

$V_d(R) = \frac{\pi^{d/2}}{(d/2)!} R^d$ and $\frac{1}{2}! = \frac{\sqrt{\pi}}{2} \rightarrow \frac{3}{2}! = \frac{3\sqrt{\pi}}{2 \cdot 2}$

3N-dim. with $R = \sqrt{2mE}$

(Boltzmann) → $S \approx k_B \log \Omega(E,N,V)$

$$P = T \frac{\partial S}{\partial V}\Big|_E = T \frac{\partial}{\partial V} k_B (\log V^N + \ldots)\Big|_E$$

$$= T k_B \frac{N}{V}$$ → ideal gas law

$$S \approx k_B \log \left[\frac{V^N}{N^N e^{-N}} \left(\frac{2\pi m E}{\frac{3N}{2} h^2} \right)^{3/2} \frac{1}{e^{-3N/2}} \right]$$

using Stirling : $N! \approx N^N e^{-N}$

$\left(\frac{3N}{2}\right)! \approx \left(\frac{3N}{2}\right)^{3N/2} e^{-3N/2}$

$$= N k_B \log \left[\frac{V}{N} \left(\frac{4\pi m E}{3 N h^2} \right)^{3/2} \right] + \frac{5N}{2} k_B$$

$e^{-N} e^{-3N/2} = e^{-5N/2}$

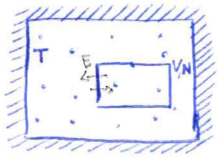
→ extensivity shows that $S(\alpha E, \alpha V, \alpha N) = \alpha S(E,V,N)$

origin extensivity → N! → QM origin (indistinguishability particles)

does not depend on size or amount syst.

intensive		extensive	
P	⟷	V	applied P adjusts V $\frac{1}{V}$
μ	⟷	N	
T	⟷	S	
…		E …	

canonical ensemble



system in contact with a thermal bath fixed at temp T

N, V also fixed

system and bath can exchange energy E → no longer fixed

$$\rightarrow \text{prob. dens}: \rho_c(\Gamma) = \frac{e^{-\beta \mathcal{H}(\Gamma)}}{N!h^{3N} \int \frac{d\Gamma}{N!h^{3N}} e^{-\beta\mathcal{H}(\Gamma)}} = \frac{e^{-\beta\mathcal{H}(\Gamma)}}{N!h^{3N} Z(N,V,T)}$$

$\beta = \frac{1}{k_B T}$

$T(s) = \int_0^\infty f(t) e^{-st} dt$ → laplace transfo of $f(t)$

with $Z(N,V,T) = \int \frac{d\Gamma}{N!h^{3N}} e^{-\beta\mathcal{H}(\Gamma)}$ the canonical partition function

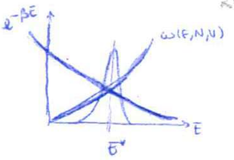
Laplace transfo ↳ $Z(\beta) = \int_0^\infty e^{-\beta E} g(E) dE$

$$= \int dE \, e^{-\beta E} \omega(E,N,V)$$

since $\int_{-\infty}^{+\infty} dE \, \delta(E - \mathcal{H}(\Gamma)) = 1$ (before $\int d\Gamma \delta(E - \mathcal{H}(\Gamma))$)

$\rightarrow Z = \int dE \int \frac{d\Gamma}{N!h^{3N}} e^{-\beta\mathcal{H}(\Gamma)} \delta(E - \mathcal{H}(\Gamma))$

$= \int dE \, e^{-\beta E} \underbrace{\int \frac{d\Gamma}{N!h^{3N}} \delta(E - \mathcal{H}(\Gamma))}_{\omega(E)}$



now for N,V,T → ∞ (TD limit)

$\omega(E,N,V)$ fast growing function E

$e^{-\beta E}$ fast decreasing function E

→ peaked around char. value E^*

or use saddle point approx (see next page)

$$\Rightarrow Z(N,V,T) \approx e^{-\beta E^*} \omega(E^*,N,V)$$

$$= e^{-\beta E^*} e^{S(E^*,N,V)/k_B}$$) Boltzmann assumption

$$= e^{-\beta[E^* - TS(E^*,N,V)]} = e^{-\beta F(E^*,N,V)}$$

$$\Rightarrow F = E - TS \approx -k_B T \log Z$$ (Helmholtz free energy)

now for non-interacting particles

$$\mathcal{H}(\Gamma) = \Sigma_{i=1}^N \mathcal{H}_1(\Gamma_i) = \Sigma_{i=1}^N \left(\frac{\vec{p}_i^{\,2}}{2m} + \varphi(\vec{q}_i) \right)$$

$$\Longrightarrow Z(N,V,T) = \frac{\{Z_1(N,V,T)\}^N}{N!}$$ with $Z_1 = \int d\vec{p} d\vec{q} \, e^{-\beta\mathcal{H}_1(\vec{p},\vec{q})}$

and in general, since the momenta integrals are gaussian integrals

Gaussian integr. $\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}$

$\int_{-\infty}^{+\infty} e^{-\frac{1}{k_B T}\frac{p^2}{2m}} dp = \sqrt{2\pi m k_B T}$

$$Z(N,V,T) = \frac{1}{N! \lambda_T^{3N}} \int d\vec{q}_1 \ldots d\vec{q}_N \, e^{-\beta\varphi(\vec{q}_1, \ldots, \vec{q}_N)} \equiv \frac{Q(N,V,T)}{N!\lambda_T^{3N}}$$

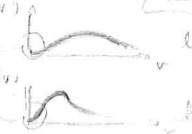
→ kinetic energy

with $Q(N,V,T)$ the configurational partition function

and $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$ the thermal wavelength

$h(v) = \int d\vec{v}\, p(\vec{v})\, \delta(|\vec{v}|-v)$
$= \dots v^2 e^{-mv^2/2k_BT}$

from jacobian $dv_x dv_y dv_z = 4\pi r^2 dr$ →

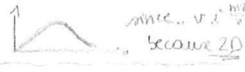


light particle

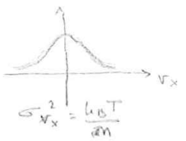
heavy part

probability particle with speed v

in simulations: typical



since .. $v\, e^{-\frac{mv^2}{2k_BT}}$
because 2D



$\sigma^2_{v_x} = \frac{k_BT}{m}$

momenta gaussian distr ⇒ averages $\langle A(\vec{p})\rangle = \langle A(\vec{v})\rangle$ trivial
- distr. velocities follows a universal law indep. on type of interaction

* MAXWELL DISTRIBUTION OF VELOCITIES

$\dots \prod_{i=1}^{3} \frac{p_i^{?}}{2m} \int d\vec{p}_i\, e^{-\beta \sum_{i=1}^{N} \vec{p}_i^2} \int d\vec{q}_1 \dots d\vec{q}_N e^{-\beta\varphi}$
$= \dots e^{-\beta p^2/2m}$

$$p(\vec{v}) = \left(\frac{m}{2\pi k_BT}\right)^{3/2} e^{-m\vec{v}^2/2k_BT}$$ (prob distr.)

$$\Rightarrow E = \langle \mathcal{H}(\Gamma)\rangle = -\frac{\partial \log Z(N,V,T)}{\partial\beta}$$

internal energy
or average energy

$$\hookrightarrow \langle A\rangle = \int d\Gamma\, \rho(\Gamma)\, A(\Gamma) = \frac{\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)} A(\Gamma)}{\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)}}$$

with $\rho(\Gamma)$ norm.: $\int d\Gamma \rho(\Gamma) = 1$

$$\rightarrow \langle\mathcal{H}\rangle = \frac{\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)} \mathcal{H}(\Gamma)}{\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)}} = -\frac{\partial}{\partial\beta} \log\left(\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)}\right)$$

since $\frac{\partial}{\partial\beta} \log\left(\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)}\right) = \frac{1}{\int d\Gamma e^{-\beta\mathcal{H}(\Gamma)}} \int d\Gamma (-\mathcal{H}) e^{-\beta\mathcal{H}(\Gamma)}$

and $Z(N,V,T) = \int \frac{d\Gamma}{N!\, h^{3N}} e^{-\beta\mathcal{H}(\Gamma)}$

thus $\langle\mathcal{H}\rangle = -\frac{\partial}{\partial\beta} \log Z$

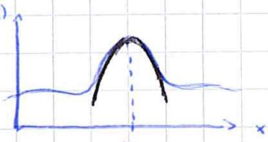
- saddle point approx

to go from $Z(N,V,T) = \int dE\, e^{-\beta E} \omega(E,N,V)$ to $Z(N,V,T) \approx e^{-\beta E_*} \omega(E_*,N,V)$
approx as follows

approx with a parabola

$$I = \int dx\, e^{Nf(x)} \approx e^{Nf(x_0)} \int_{-\infty}^{+\infty} dx\, e^{\frac{N}{2} f''(x_0)(x-x_0)^2} \xrightarrow{N\to\infty} = \underbrace{e^{Nf(x_0)}}_{\text{dominant part}} \sqrt{\frac{2\pi}{N|f''(x_0)|}}$$

$f(x) \approx f(x_0) - \frac{A}{2}(x-x_0)^2$
$+ \dots$



> dom. part is the maximum

$\rightarrow \log I = Nf(x_0) + O(\log N)$
$\Rightarrow$ taking $\log Z = \beta(E-TS)$ is a good approx.

- thermal wavelength

$\lambda_T = h/\sqrt{2\pi m k_BT}$ ~ average de Broglie wavelength gas part in ideal gas at specified T
→ the Broglie wavelength for a part. with average thermal energy
$= h/p_T$

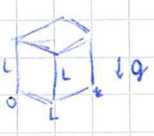
$p_T = \sqrt{\frac{2mE}{N}} = \sqrt{3mk_BT}$
= average mom per part

comparing the wavelength with the typical distance betw. part
$\Rightarrow$ if $\lambda_T \ll l$ → classical regime
if $\lambda_T \gtrsim l$ → quantum regime

- non-interacting particles

$$Z(N,V,T) = \frac{1}{N!\,\lambda_T^{3N}} \int d\vec{q}_1 \dots d\vec{q}_N\, e^{-\beta\varphi(\vec{q}_1)} e^{-\beta\varphi(\vec{q}_2)} \dots e^{-\beta\varphi(\vec{q}_N)}$$
$$= \frac{1}{N!\,\lambda_T^{3N}} \int d\vec{q}_1 e^{-\beta\varphi(\vec{q}_1)} \int d\vec{q}_2 e^{-\beta\varphi(\vec{q}_2)} \dots$$
$$= \frac{1}{N!\,\lambda_T^{3N}} \left[\int d\vec{q}\, e^{-\beta\varphi(\vec{q})}\right]^N = \frac{[Q(1,V,T)]^N}{N!\,\lambda_T^{3N}}$$

→ in a gravitational field



$$Q(N,V,T) = \int d\vec{q}_1 \dots d\vec{q}_N\, e^{-\beta mg(z_1+z_2+\dots+z_N)}$$
$$= \int dx_1 dy_1 dz_1\, e^{-\beta mgz_1} \int dx_2 dy_2 dz_2\, e^{-\beta mgz_2} \dots$$
$$= \left[\int_0^L dx \int_0^L dy \int_0^L dz\, e^{-\beta mgz}\right]^N = \left[L^2 \frac{1}{\beta mg}(1-e^{-\beta mgL})\right]^N$$

example: Ideal gas

$$Z = \frac{1}{N!\,h^{3N}} \int d\Gamma\, e^{-\beta[\sum_i \vec{p}_i^2/2m + \sum_i \varphi_w(\vec{q}_i)]} = \frac{1}{N!\,h^{3N}} \left[\int d\vec{p}\, d\vec{q}\, e^{-\beta(\frac{\vec{p}^2}{2m} + \varphi_w(\vec{q}))}\right]^N$$

(same φ_{wall} as before)

$$= \frac{V^N}{N!\,h^{3N}} \left[\int d\vec{p}\, e^{-\beta\vec{p}^2/2m}\right]^N = \frac{V^N}{N!\,h^{3N}} \left[\int dp\, e^{-\beta p^2/2m}\right]^{3N} = \frac{V^N}{N!\,h^{3N}} (2m\pi/\beta)^{3N/2}$$

$$= \frac{V^N}{N!\,\lambda_T^{3N}}$$

$\mathcal{Z} = \frac{V^N}{N!\lambda_T^{3N}}$ $\rightarrow$ $F = -k_BT\log \mathcal{Z} = -k_BT[N\log V - \underline{3N\log\lambda_T} - \log N!]$

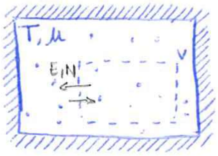
$P = -\frac{\partial F}{\partial V}\Big|_{N,T} = \frac{Nk_BT}{V}$ $\rightarrow$ ideal gas law

$\lambda_T = \frac{h}{\sqrt{2\pi m k_BT}} = A\,\beta^{1/2}$

and $E = \langle H\rangle = -\frac{\partial \log Z}{\partial\beta} = \frac{\partial\, 3N\log\beta^{1/2}}{\partial\beta} = \frac{3N}{2}\frac{\partial\log\beta}{\partial\beta} = \frac{3}{2}Nk_BT$ OK!

= average kinetic energy of a system

Grandcanonical ensemble



V, T, μ (chemical potential) fixed
E, N can fluctuate

$\rightarrow$ prob dens $\rho_{gc}(\Gamma_N, N) = \frac{e^{-\beta\mathcal{H}_N(\Gamma_N)}\,e^{\beta\mu N}}{N!\,h^{3N}\,\Xi(\mu,V,T)}$

$\sum_N\int d\Gamma_N\,\rho_{gc}(\Gamma_N,N) = 1$

with $\Xi(\mu,V,T) = \sum_N e^{\beta\mu N}\underbrace{\int\frac{d\Gamma_N}{N!\,h^{3N}}e^{-\beta\mathcal{H}(\Gamma_N)}}_{\mathcal{Z}(N,V,T)}$

Laplace transform

$= \sum_N e^{\beta\mu N}\mathcal{Z}(N,V,T) = \sum_N e^{\beta\mu N}\int dE\, e^{-\beta E}\,\omega(E,N,V)$

sum dominated by N* the most probable value of the # of particles

$\approx e^{\beta\mu N_*}\mathcal{Z}(N_*,V,T)$

$= e^{\beta\mu N_*}e^{-\beta F} = e^{-\beta[-\mu N_* + E - TS]} = e^{\beta PV}$

where
$E = TS - PV + \mu N$
$\rightarrow \mu N = E - TS + PV = G$
(Euler relation)

$\Rightarrow k_BT\log\Xi = PV$

we can also show that

$E = \langle\mathcal{H}(\Gamma)\rangle = -\frac{\partial\log\Xi(N,V,T)}{\partial\beta}\Big|_{\beta\mu=cts}$

$\langle\mathcal{H}(\Gamma)\rangle = \frac{\sum_N \mathcal{H}(\Gamma_N)e^{\beta\mu N}Z(N,V,T)}{\sum_N e^{\beta\mu N}Z(N,V,T)} = -\frac{\partial\log Z}{\partial\beta} = -\frac{\partial\log\Xi}{\partial\beta}\Big|_{\beta\mu=cts}$

$\langle N\rangle = \frac{\partial\log\Xi(N,V,T)}{\beta\,\partial\mu}$

$\langle N\rangle = \frac{\sum_N Ne^{\beta\mu N}Z(N,V,T)}{\underbrace{\sum_N e^{\beta\mu N}Z(N,V,T)}_{\Xi}} = \frac{1}{\beta}\frac{\partial\log\Xi}{\partial\mu} = \frac{1}{\beta\Xi}\frac{\partial\Xi}{\partial\mu}$ OK!

example: Ideal gas

$\Xi = \sum_N e^{\beta\mu N}\mathcal{Z}(N,V,T) = \sum_N e^{\beta\mu N}\frac{V^N}{N!\lambda_T^{3N}} = \sum_N\left(\frac{e^{\beta\mu}V}{\lambda_T^3}\right)^N\frac{1}{N!}$

$= \exp\left(\frac{e^{\beta\mu}V}{\lambda_T^3}\right)$ $\lambda_T \sim \beta^{1/2}$ $\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$\Rightarrow \frac{PV}{k_BT} = \log\Xi = \frac{e^{\beta\mu}V}{\lambda_T^3}$ $\rightarrow$ how does this relate to the ideal gas law $PV = Nk_BT$?

$\langle N\rangle = \frac{\partial\log\Xi}{\beta\partial\mu} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\left(\frac{e^{\beta\mu}V}{\lambda_T^3}\right) = \frac{e^{\beta\mu}V}{\lambda_T^3}$ $\longrightarrow$ $\frac{PV}{k_BT} = \langle N\rangle$ OK!

$E = -\frac{\partial\log\Xi}{\partial\beta} = -\frac{\partial}{\partial\beta}\left(\frac{e^{\beta\mu}V}{\lambda_T^3}\right)\Big|_{\beta\mu=cts} = -e^{\beta\mu}V\frac{\partial}{\partial\beta}\left(\sqrt{\frac{2\pi m}{\beta h^2}}^3\right) = \frac{3}{2}e^{\beta\mu}V\sqrt{\frac{2\pi m}{h^2}}^3\sqrt{\beta}^{-5}$

$= \frac{3}{2}\frac{1}{\beta}\frac{e^{\beta\mu}V}{\lambda_T^3} = \frac{3}{2}k_BT\langle N\rangle$

$\rightarrow$ Ideal gas

syst with non-interacting particles : $\mathcal{H} = \sum_{i=1}^N\frac{\vec{p}_i^2}{2m} + \sum_{i=1}^N\varphi_W(\vec{q}_i)$

↳ with confining potential $\varphi_W(\vec{q}_i) = \begin{cases}0 & \text{if } \vec{q}_i\in V\\ +\infty & \text{else}\end{cases}$

micro-canonical $\rightarrow \omega(E,N,V) = \frac{V^N}{N!\,h^{3N}}2\pi m\frac{(2\pi mE)^{3N/2-1}}{(\frac{3N}{2}-1)!}$

$\omega(E,N,V) = \frac{V^N}{N!h^{3N}}\int d\vec{p}_1\ldots d\vec{p}_N\,\delta\left(E - \sum_{i=1}^N\frac{\vec{p}_i^2}{2m}\right)$

$= \frac{V^N}{N!\,h^{3N}}\mu_{3N-1}\int_0^\infty dp\,p^{3N-1}\delta\left(E - \frac{p^2}{2m}\right)$

$= \frac{V^N}{N!\,h^{3N}}$ TODO

"polar" coord $d^dx = \mu_{d-1}x^{d-1}dx$ with $\mu_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

Stirling $N! \approx e^{N}N^N$

$\sim\left(\frac{V}{N}\right)^N\left(\frac{4\pi mE}{3h^2N}\right)^{3N/2}e^{5N/2}$

$\rightarrow p_{mc}(\vec{p}) = \langle\delta(\vec{p}_1 - \vec{p})\rangle_{mc}$

$\sim \ldots\exp\left(-\frac{3N\vec{p}^2}{4mE}\right)$ || TODO $\rightarrow$ p41

canonical $\rightarrow Z(N,V,T) = \frac{V^N}{N!\,\lambda_T^{3N}}$

$\rightarrow p_c(\vec{p}) = \langle\delta(\vec{p} - \vec{p}_1)\rangle_c = \ldots e^{-\beta\frac{\vec{p}^2}{2m}}$ || TODO $\rightarrow$ p 42

$\Rightarrow$ you'll see that with the 3 ensembles, the ideal gas law $PV = Nk_BT$ and the internal energy $E = \frac{3}{2}Nk_BT$ can be calculated (see throughout summary)

} if $N\rightarrow\infty$: mc, c and Gc all become the same

Equipartition theorem

- $\left\langle x_r \frac{\partial \mathcal{H}}{\partial x_s} \right\rangle = k_B T \delta_{rs}$, with x_r a comp. of pos. or mom. of the particle
  ↳ thermal average

$\rightarrow \langle q_{i,x} \partial_{q_{i,y}} \mathcal{H} \rangle = \langle q_{i,x} \partial_{q_{j,x}} \mathcal{H} \rangle = \langle p_{i,x} \partial_{p_{j,x}} \mathcal{H} \rangle = 0$ for $i \neq j$

now

$$\left\langle \vec{p}_i \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}_i} \right\rangle = \left\langle p_{i,x} \frac{\partial \mathcal{H}}{\partial p_{i,x}} \right\rangle + \left\langle p_{i,y} \frac{\partial \mathcal{H}}{\partial p_{i,y}} \right\rangle + \left\langle p_{i,z} \frac{\partial \mathcal{H}}{\partial p_{i,z}} \right\rangle = 3k_B T$$

$\Rightarrow$ total kinetic energy (summing over all $\overset{N}{\text{part.}}$): $E_{kin} = \frac{3Nk_BT}{2}$

since $\left\langle \vec{p} \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}} \right\rangle = \left\langle \vec{p} \cdot \frac{\vec{p}}{m} \right\rangle = 2 \left\langle \frac{p^2}{2m} \right\rangle = 2 \langle \mathcal{H} \rangle$.

↳ since $E_{kin} = E_{ideal\ gas}$ we can conclude that the only contribution comes from the kinetic energy

and

$$\left\langle \vec{q}_i \cdot \frac{\partial \mathcal{H}}{\partial \vec{q}_i} \right\rangle = \left\langle q_{i,x} \frac{\partial \mathcal{H}}{\partial q_{i,x}} \right\rangle + \left\langle q_{i,y} \frac{\partial \mathcal{H}}{\partial q_{i,y}} \right\rangle + \left\langle q_{i,z} \frac{\partial \mathcal{H}}{\partial q_{i,z}} \right\rangle = 3k_B T$$

$$= \left\langle \vec{q}_i \cdot \frac{\partial \varphi}{\partial \vec{q}_i} \right\rangle = - \langle \vec{q}_i \cdot \vec{F}_i \rangle$$

$\Rightarrow \sum_{i=1}^{N} \langle \vec{q}_i \cdot \vec{F}_i \rangle = -3Nk_BT$ with $\vec{F}_i$ the total force applied to part i

proof of the theorem

$$\left\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \right\rangle = \frac{\int d\Gamma\, x_i \frac{\partial \mathcal{H}}{\partial x_j} e^{-\beta \mathcal{H}}}{\underbrace{\int d\Gamma\, e^{-\beta \mathcal{H}}}_{=\mathcal{Z}}} = \frac{-1}{\beta \mathcal{Z}} \int d\Gamma\, x_i \frac{\partial}{\partial x_j} \left(e^{-\beta \mathcal{H}} \right)$$

$$\frac{\partial}{\partial x_j}(x_i e^{-\beta \mathcal{H}}) = \frac{\partial x_i}{\partial x_j} e^{-\beta \mathcal{H}} + x_i \frac{\partial e^{-\beta \mathcal{H}}}{\partial x_j}$$

$$= \frac{-1}{\beta \mathcal{Z}} \int d\Gamma \left[\underbrace{\frac{\partial}{\partial x_j}(x_i e^{-\beta \mathcal{H}})}_{=0} - e^{-\beta \mathcal{H}} \underbrace{\frac{\partial x_i}{\partial x_j}}_{=\delta_{ij}} \right]$$

$$\int d\Gamma \frac{\partial}{\partial x_j}(x_i e^{-\beta \mathcal{H}}) = \int d\Gamma_j (x_i e^{-\beta \mathcal{H}}) \Big|^{x_j} = 0$$

$$= \frac{+1}{\beta \mathcal{Z}} \int d\Gamma\, e^{-\beta \mathcal{H}} \delta_{ij}$$

$$= k_B T \delta_{ij} \quad \square$$

$$\int dp_{1,x} dp_{1,y} \dots dp_{k,y} \dots dq_{1,x} \dots dq_{N,z} \left[\frac{\partial}{\partial p_{k,y}} (x_i e^{-\beta \mathcal{H}}) \right]$$

partial integration
$$= \int \underbrace{dp_{1,x} dp_{1,y} \dots dq_{1,x} \dots dq_{N,z}}_{6N-1} \left[x_i e^{-\beta \mathcal{H}} \Big|_{p_{k,y}=+\infty} - x_i e^{-\beta \mathcal{H}} \Big|_{p_{k,y}=-\infty} \right]$$

with $e^{-\beta \mathcal{H}} \sim e^{-p_{k,y}^2/2m}$

$\Rightarrow 0$

for the potential: $\varphi \rightarrow +\infty$ at boundaries since there is a wall confining the part within volume V

Diatomic molecules

$\underset{1}{o}\, m\, \underset{2}{o}$ $\quad \mathcal{H}_{mol} = \frac{\vec{p}_1}{2m} + \frac{\vec{p}_2}{2m} + \frac{k}{2} |\vec{q}_1 - \vec{q}_2|^2$

$$\mathcal{Z}(N,V,T) = \frac{\mathcal{Z}_{mol}(V,T)^N}{N!}$$

$$\rightarrow \mathcal{Z}_{mol}(V,T) = \frac{1}{h^6} \int d\vec{p}_1 d\vec{p}_2 d\vec{q}_1 d\vec{q}_2\, e^{-\beta[\vec{p}_1/2m + \vec{p}_2/2m + k/2|\vec{q}_1 - \vec{q}_2|^2]}$$

$$= \frac{1}{\lambda_T^6} \int d\vec{q}_1 d\vec{q}_2\, e^{-\beta k/2\, |\vec{q}_1 - \vec{q}_2|^2} \quad ; \quad \vec{q} = \vec{q}_1 - \vec{q}_2, \quad \vec{Q}_{cm} = \frac{\vec{q}_1 + \vec{q}_2}{2}$$

$$= \frac{1}{\lambda_T^6} \int d\vec{Q}_{cm} d\vec{q}\, e^{-\beta k/2\, \vec{q}^2}$$

$$= \frac{V}{\lambda_T^6} \int d\vec{q}\, e^{-\beta k \vec{q}^2/2} = \frac{V}{\lambda_T^6} \left(\frac{2\pi}{\beta k} \right)^{3/2}$$

$$\rightarrow F = -k_BT \log \mathcal{Z} = -k_BT \cdot \log \frac{\mathcal{Z}_{mol}^N}{N!} \approx -Nk_BT \log \mathcal{Z}_{mol} + \cancel{N} k_BT (N \log N - N)$$

$\hookrightarrow N! \approx e^N N^N$

$\log N! \approx N \log N - N$

$$= -Nk_BT \log \frac{V}{N\lambda_T^6} \left(\frac{2\pi}{\beta k} \right)^{3/2} - Nk_BT$$

$\rightarrow C_v = \frac{\partial E}{\partial T}\Big|_{V,N}$, $E = \langle \mathcal{H} \rangle = -\frac{\partial \log \mathcal{Z}}{\partial \beta}$

$$\mathcal{Z}_{mol} \sim \frac{1}{\lambda_T^6} \frac{1}{\beta^{3/2}} \sim (\beta^{1/2})^{-6} \frac{1}{\beta^{3/2}} \sim \overset{\sim\text{momentum}}{\frac{1}{\beta^3}} \overset{\sim\text{spring}}{\frac{1}{\beta^{3/2}}} \sim \frac{1}{\beta^{9/2}} \rightarrow \log \mathcal{Z}_{mol} = \dots + \log \beta^{9/2}$$

$\rightarrow E_{mol} = \frac{9}{2} k_BT \quad \rightarrow \quad C_{v,mol} = \frac{9}{2} k_B$

$\rightarrow \langle |\vec{q}_1 - \vec{q}_2|^2 \rangle = \langle |\vec{q}^2| \rangle$

$\hookrightarrow \mathcal{H}_{mol} = \dots + \frac{\vec{q}^2}{2} k \rightarrow$ equip $\left\langle \vec{q} \cdot \frac{\partial \mathcal{H}}{\partial \vec{q}} \right\rangle = \langle \vec{q}^2 k \rangle = 3k_BT \rightarrow \langle \vec{q}^2 \rangle = \frac{3k_BT}{k}$

and $\langle \vec{q}^2 \rangle = \frac{-2}{\beta} \frac{\partial \cancel{\log \mathcal{Z}}}{\partial k} \log \mathcal{Z}_{mol}$; $\mathcal{Z}_{mol} \sim k^{-3/2}$

$$= \frac{2}{\beta} \frac{3}{2} \frac{1}{k} = \frac{3k_BT}{k}$$

Energy fluctuations of ideal gases

Describing fluctuations: σ_a for quantity a

$\frac{\sigma_a^2}{N}$: how they change with N ; $\frac{\sigma_a}{\langle a \rangle}$: relative fluct.

$$\sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2 = \frac{\partial^2 \log Z}{\partial\beta^2}$$

proof $Z = \int d\Gamma\, e^{-\beta H(\Gamma)} = \sum e^{-\beta E}$

$$\rightarrow -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \left(-\frac{\partial \log Z}{\partial\beta} =\right) \langle E\rangle \qquad -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{-1}{Z}\int d\Gamma\,(-H(\Gamma))\, e^{-\beta H(\Gamma)} = \langle H(\Gamma)\rangle$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} = \langle E^2\rangle \qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} = \frac{1}{Z}\int d\Gamma\,(-H(\Gamma))^2 e^{-\beta H(\Gamma)} = \langle (H(\Gamma))^2\rangle$$

$$\frac{\partial^2 \log Z}{\partial\beta^2} = \frac{\partial}{\partial\beta}\left[\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right] = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial\beta}\right)^2$$

$$= \langle E^2\rangle - \langle E\rangle^2$$

$$\sigma_N^2 = \langle N^2\rangle - \langle N\rangle^2 = \frac{\partial^2 \log \Xi}{\beta^2 \partial\mu^2}$$

proof $\Xi = \sum_N e^{\beta\mu N} Z(N,V,T)$

$$\rightarrow +\frac{\partial \log \Xi}{\beta\partial\mu} = \frac{1}{\beta\Xi}\frac{\partial\Xi}{\partial\mu} = \langle N\rangle \qquad \frac{1}{\beta\Xi}\frac{\partial\Xi}{\partial\mu} = \frac{1}{\beta\Xi}\sum_N \beta N e^{\mu\beta N} Z(N,V,T) = \langle N\rangle$$

$$\frac{1}{\beta^2\Xi}\frac{\partial^2\Xi}{\partial\mu^2} = \langle N^2\rangle \qquad \frac{1}{\beta^2\Xi}\frac{\partial^2\Xi}{\partial\mu^2} = \langle N^2\rangle$$

$$\frac{\partial^2 \log\Xi}{\beta^2\partial\mu^2} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\left[\frac{1}{\beta\Xi}\frac{\partial\Xi}{\partial\mu}\right] = \frac{1}{\beta^2\Xi}\frac{\partial^2\Xi}{\partial\mu^2} - \frac{1}{\beta^2\Xi^2}\left(\frac{\partial\Xi}{\partial\mu}\right)^2 = \langle N^2\rangle - \langle N\rangle^2$$

→ for an ideal gas

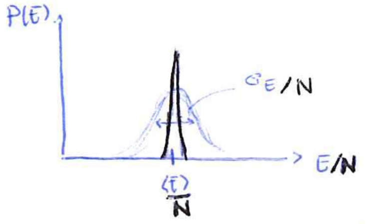
$$Z = \frac{V^N}{N!\lambda_T^3} \sim \beta^{-3N/2}$$

$$\Rightarrow \log Z = \dots - \frac{3N}{2}\log\beta$$

$$\rightarrow \frac{\partial^2 \log Z}{\partial\beta^2} = \frac{3N}{2}\frac{1}{\beta^2} \Rightarrow \sigma_E^2 = \frac{3N}{2}(k_BT)^2$$

$$\rightarrow \frac{\sigma_E^2}{N} \sim \frac{1}{\sqrt{N}}$$

and $\langle E\rangle = \frac{3Nk_BT}{2}$



→ N larger ⇒ fluctuations smaller and smaller

$$N\rightarrow\infty \; : \frac{\sigma_E}{\langle E\rangle} \longrightarrow 0 \qquad \frac{\sqrt{\frac{3N}{2}}(k_BT)}{\frac{3N}{2}k_BT} = \frac{1}{\sqrt{3N/2}}$$

→ ensembles become equivalent, canonical → microcanonical

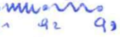
Harmonic oscillator in 1D

$$\varphi(\vec{q}) = \frac{k}{2}\vec{q}^2$$

$$\rightarrow Z = \dots \int dq\, e^{-\frac{\beta k}{2}q^2}$$

$$= \dots \frac{1}{\sqrt{\beta k}}\int = \dots \frac{1}{\beta^{1/2}}$$ ⇒ only look at contribution oscillation

$$\rightarrow E = \frac{k_BT}{2}$$

for 2 oscillations


$$Z = \dots \frac{1}{\sqrt{\beta k_1}}\frac{1}{\sqrt{\beta k_2}}\int\dots \rightarrow \langle E\rangle = \frac{k_BT}{2} + \frac{k_BT}{2}$$

+ EQUIPARTITION THEOREM

every quadratic degree of freedom contributes as $\frac{k_BT}{2}$

$$\Rightarrow \text{for } H_{mol} = \frac{\vec{p}_1}{2m} + \frac{\vec{p}_2}{2m} + \frac{k}{2}|\vec{q}_1 - \vec{q}_2|^2$$

↓ 3 quadr dof ↘ 3 ↘ 3

$$\Rightarrow \frac{9k_BT}{2}$$

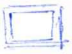
carefull for situations like these:

$\varphi(q) = \frac{k}{2}q^2 + \frac{k'}{4}q^4$ → can't apply equip.

but $\varphi(q) = \frac{k'}{4}q^4$ → equip: $E = \frac{k_BT}{4}$

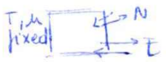
---

recap

$\rho_{mc}(\Gamma) = \dots \delta(E - H(\Gamma))$, E,N,V

E fixed

$\rho_c(\Gamma) = \dots e^{-\beta H(\Gamma)}$, T,N,V


– "preferred" ensemble in CSM

$\rho_{uc}(\Gamma) = \dots e^{-\beta H(\Gamma)} e^{\beta\mu N}$, T,μ,V


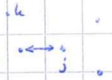
– used ensemble in QSM

$\boxed{3}$ Interacting Systems

so far we have used $Z(N,V,T)=\frac{1}{N!}[Z(1,V,T)]^N$
however this is NOT valid if we have interactions! (interactions ⇝ potentials)

Pair potentials

$$\mathcal{H}(\Gamma)=\sum_{i=1}^{N}\vec{p}_i^{\,2}/2m+\Phi(\vec{q}_1,\ldots,\vec{q}_N)$$

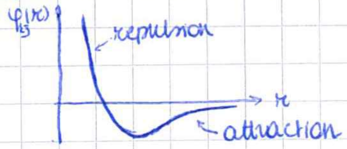


→ pairwise interactions: $\Phi=\sum_{i<j}\varphi(|\vec{q}_i-\vec{q}_j|)$

thus $Q(N,V,T)=\int d\vec{q}_1\ldots d\vec{q}_N\, e^{-\beta\sum_{i<j}\varphi_{ij}}$

$=\int d\vec{q}_1\ldots d\vec{q}_N\, e^{-\beta\varphi_{12}}e^{-\beta\varphi_{13}}\ldots e^{-\beta\varphi_{23}}\ldots e^{-\beta\varphi_{N-1,N}}$

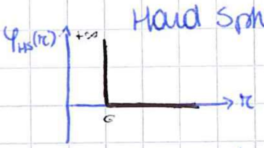
example Lennard-Jones potential



$$\varphi_{LJ}(r)=\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12}-\left(\frac{\sigma}{r}\right)^{6}\right]$$

repulsive at short distances < strong repulsive energy against overlapping of e⁻ orbitals
attractive at longer distances < van der Waals interactions between neutral atoms

Hard Spheres potential

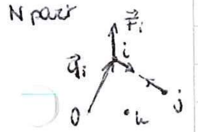


$$\varphi_{HS}(r)=\begin{cases}+\infty & 0\leq r<\sigma\\ 0 & r\geq\sigma\end{cases}$$ → short distance repulsion (only)

↳ since $Z(N,V,T)=\frac{1}{N!\lambda_T^{3N}}\int d\vec{q}_1\ldots d\vec{q}_N\, e^{-\beta\sum_{i<j}\varphi_{ij}}$ cannot be computed as Φ cannot be factorized
⇒ approximated techniques useful if SYSTEM IS DILUTED ($\underline{n}$ low)
should include hard wall pot.
→ virial expansion: P as an expansion in powers of $n=N/V$

$$P=nk_BT(1+\underline{b_2n+b_3n^2+\ldots})$$
ideal gas   mutual interactions between particles

Virial theorem



should always include hard wall potential in Z

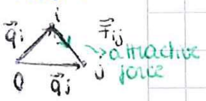
$\vec{F}_i=\vec{F}_i^{(w)}+\sum_{j\neq i}\vec{F}_{i,j}$ → total force = force from collisions with the wall of the container + force from intermolecular interactions

$\frac{\partial\varphi_{ij}}{\partial q_{i,\alpha}}=\frac{\partial\varphi_{ij}}{\partial q_{ij}}\frac{\partial q_{ij}}{\partial q_{i,\alpha}}=\frac{\partial\varphi_{ij}}{\partial q_{ij}}\frac{q_{i,\alpha}-q_{j,\alpha}}{q_{ij}}$

$\vec{F}_{ij}\equiv-\frac{\partial\varphi(|\vec{q}_i-\vec{q}_j|)}{\partial\vec{q}_i}=-\frac{\partial\varphi(q_{ij})}{\partial q_{ij}}\frac{\vec{q}_i-\vec{q}_j}{q_{ij}}$ force that j exerts on i

$\left.\begin{aligned}|\vec{q}_i-\vec{q}_j|&=q_{ij}\\ \varphi_{ij}&=\varphi(q_{ij})\end{aligned}\right\}$

→ equip: $\sum_{i=1}^{N}\langle\vec{q}_i\cdot\vec{F}_i^{(w)}\rangle+\sum_{i,j=1}^{N}\langle\vec{q}_i\cdot\vec{F}_{ij}\rangle=-3Nk_BT$ with $\vec{F}_{ii}=0$



force perpendicular to wall, pointing inwards

* $\vec{F}^{(w)}=\frac{\Delta\vec{p}}{\Delta t}=\frac{-2\vec{p}_\perp}{\Delta t}$ particles collide elastically with the wall

now $\vec{p}=\vec{p}_\parallel+\vec{p}_\perp$ (∥ to wall, ⊥ to wall momentum)

$\vec{p}\,'=-\vec{p}_\perp+\vec{p}_\parallel$ after collision

contr. from specific location

→ $\sum_{i=1}^{N}\langle\vec{q}_i\cdot\vec{F}_i^{(w)}\rangle_{\vec{r},dS}$ dS portion of wall @ point $\vec{r}$

$=\vec{r}\cdot\sum\langle\vec{F}_i^{(w)}\rangle_{\vec{r},dS}$ only particles in $\vec{q}_i=\vec{r}$ contribute → $\Delta\vec{p}=\vec{p}\,'-\vec{p}=\vec{p}_\parallel-\vec{p}_\perp-\vec{p}_\perp-\vec{p}_\parallel=-2\vec{p}_\perp$

$=-\vec{r}\cdot(P\,d\vec{S})$ P = pressure = average force per unit area exerted on the wall by the particles

↳ $\vec{F}^{(w)}$ points opposite to $\hat{n}$; $d\vec{S}=\hat{n}\,dS$ with $\hat{n}$ perpendicular in $\vec{x}$, pointing outwards

integr. over whole surface

→ $\sum_{i=1}^{N}\langle\vec{q}_i\cdot\vec{F}_i^{(w)}\rangle=-P\oint_S\vec{r}\cdot d\vec{S}=-P\int\vec{\nabla}\cdot\vec{r}\,dV=-3P\int_V dV=-3PV$

$\vec{\nabla}\cdot\vec{r}=dx/dx+dy/dy+dz/dz=3$

* $\sum_{i,j=1}^{N}\langle\vec{q}_i\cdot\vec{F}_{ij}\rangle=\frac{1}{2}\sum_{i,j=1}^{N}\langle\vec{q}_i\cdot\vec{F}_{ij}+\vec{q}_j\cdot\vec{F}_{ji}\rangle=\frac{1}{2}\sum_{i,j=1}^{N}\langle(\vec{q}_i-\vec{q}_j)\cdot\vec{F}_{ij}\rangle$ ($\vec{F}_{ij}=-\vec{F}_{ji}$, NIII)

$=-\frac{1}{2}\sum_{i,j=1}^{N}\langle q_{ij}\frac{d\varphi(q_{ij})}{dq_{ij}}\rangle$ ↲ $\vec{F}_{ij}=-\frac{d\varphi_{ij}}{dq_{ij}}\frac{\vec{q}_i-\vec{q}_j}{q_{ij}}$

$\frac{(\vec{q}_i-\vec{q}_j)^2}{q_{ij}}=\frac{q_{ij}^2}{q_{ij}}$

(see back of this page)

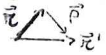
$$\sum_{i,j=1}^{N} \langle \vec{q}_i \cdot \vec{F}_{ij} \rangle = -\frac{1}{2} \sum_{i,j=1}^{N} \langle q_{ij} \frac{d\varphi(q_{ij})}{dq_{ij}} \rangle$$

$$= -\frac{1}{2} \sum_{i,j=1}^{N} \langle q_{ij} \frac{d\varphi_{ij}}{dq_{ij}} \int d\vec{r}\, d\vec{\rho}\, \delta(\vec{r}-\vec{q}_i)\delta(\vec{r}+\vec{\rho}-q_j) \rangle$$

$$= -\frac{1}{2} \int d\vec{r}\, d\vec{\rho}\, \rho \frac{d\varphi(\rho)}{d\rho} \underbrace{\langle \sum_{i=1}^{N} \delta(\vec{r}-\vec{q}_i) \sum_{j\neq i}^{N} \delta(\vec{r}+\vec{\rho}-\vec{q}_j) \rangle}_{n^{(2)}}$$

$$= -\frac{1}{2} \int d\vec{r}\, d\vec{\rho}\, \rho \frac{d\varphi(\rho)}{d\rho} n^{(2)}(\vec{r}, \vec{r}+\vec{\rho})$$

$$1 = \int d\vec{r}\, d\vec{r}\,'\, \delta(\vec{r}-\vec{q}_i)\delta(\vec{r}\,'-\vec{q}_j)$$
$$= \int d\vec{r}\, d\vec{\rho}\, \delta(\vec{r}-\vec{q}_i)\delta(\vec{r}+\vec{\rho}-\vec{q}_j)$$



for ideal gas
$n^{(2)}(\vec{r}, \vec{r}+\vec{\rho}) = n^2 = \frac{N^2}{V^2}$
$\rightarrow g(\rho) = 1$

for an ideal gas in a grav. field
$n^{(2)}(\vec{r}, \vec{r}+\vec{\rho}) = \ldots e^{-\beta m g z} e^{-\beta m g (z+\rho_z)}$

with $n^{(2)}(\vec{r}, \vec{r}+\rho)$ the pair correlation function
= prob of finding 2 particles in the volume elements $d\vec{r}$ and $d\vec{r}\,'$ around $\vec{r}$ and $\vec{r}\,'$
→ for a homogeneous system, far away from walls, with transl. & rot invariance:

for real system $n^{(2)}(\vec{r}, \vec{r}+\vec{\rho}) = n^2 g(\rho)$ → no $\vec{r}$-dependence due to translational invar.
with $n = N/V$, $g(\rho)$ = radial correlation function + only $|\vec{\rho}| = \rho$ dep. due to isotropic
for $\rho \to \infty$ (large distances) → prob. of finding the 2 part. factorizes (no correlation)
$n^{(2)} \to n \Rightarrow g(\rho) \to 1$

$$= -\frac{N^2}{2V} \int \rho \frac{d\varphi(\rho)}{d\rho} g(\rho)\, d\vec{\rho}$$

$\int d\vec{r}\, n^{(2)}(\vec{r}, \vec{\rho}) = V n^2 g(\vec{\rho})$

$$\Rightarrow -3Nk_BT = -3PV - \frac{N^2}{2V} \int \rho \frac{d\varphi(\rho)}{d\rho} g(\rho)\, d\vec{\rho}$$

$$\Rightarrow P = n k_B T - \frac{n^2}{6} \int r \frac{d\varphi(r)}{dr} g(r)\, d\vec{r}$$

↑ ! EXACT EXPRESSION

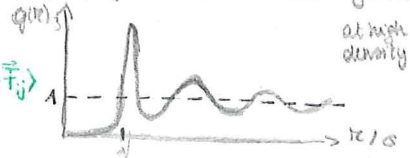
→ for non-interacting particles $\varphi = cte$
⇒ reduces to ideal gas law

↳ given by an integral over the positions of all particles
in general not solvable

$n^2 g(\rho) = \sum_{i,j} \langle \delta(\vec{r}-\vec{q}_i)\delta(\vec{r}+\vec{\rho}-\vec{q}_j) \rangle$

$PV = Nk_BT + \frac{1}{6} \sum_{ij} \langle (\vec{q}_i - \vec{q}_j) \cdot \vec{F}_{ij} \rangle$
if attractive forces "win"
$\sum \langle \ldots \rangle < 0$
$\Rightarrow PV < Nk_BT$
if repulsive forces "win"
$\sum \langle \ldots \rangle > 0$
$\Rightarrow PV > Nk_BT$

example $g(r)$: Lennard-Jones



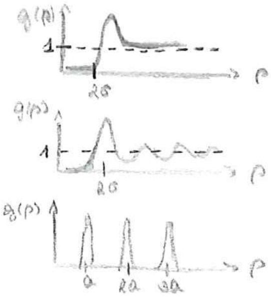
at high density

vanishes at short distances - due to repulsive part φ_{LJ}
at high ρ → damped oscillations
→ preference particles to be found @ specific distances from the origin
lower ρ → oscillations reduced
1st peak remains due to attraction, but $g(r) \to 1$ much more rapid

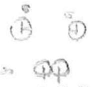
close to min φ_{LJ}
1st peak ⇒ 1st layer prevents other particles on getting close to it
→ 1st min behind peak

for ideal gas: $g(\vec{\rho}) = 1$ → goes to this value

other example



for hard spheres



→ 2σ minimal

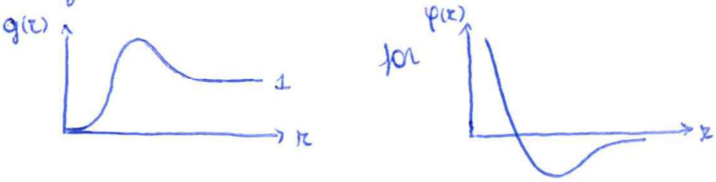
→ crystal structure

Virial expansion

radial distribution function $g(r)$ describes how the probability of finding another particle varies with r (given a fixed ref. part at $r=0$)
→ prob of finding 2nd particle

! assume DILUTE system: very low density
→ assume only one particle in its surroundings
$\Rightarrow g(r) \approx e^{-\beta\varphi(r)}$

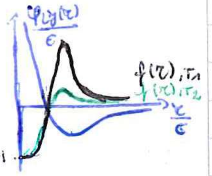
assume only 1 pair gets close



↳ for $r \to \infty$: $g(r) \to 1$ as $\varphi(r) \to 0$
neglect correlations between particles beyond pair interactions
oscillations φ_{LJ} not possible even though pair correlation since it manifests itself only at high densities

integration by parts
$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$

low n



for $g(r) \approx e^{-\beta\varphi(r)}$, we get

$$P = nk_BT - \frac{n^2}{6}\int r \frac{d\varphi(r)}{dr} g(r)\, d\vec{r} = nk_BT - \underbrace{\frac{n^2}{6}\int r \frac{d\varphi(r)}{dr} e^{-\beta\varphi(r)}\, d\vec{r}}_{\text{correction } O(n^2)}$$

$$= nk_BT + \frac{n^2}{6\beta}\int \vec{r}\cdot\frac{d}{d\vec{r}}\left(e^{-\beta\varphi(r)} - 1\right) d\vec{r}$$

[illegible] since no effect and useful later
will avoid divergences at boundaries

$$= nk_BT - \frac{n^2 k_BT}{2}\int \left(e^{-\beta\varphi(r)} - 1\right) d\vec{r}$$

integration by parts with change of coord
$d\vec{r} \to 4\pi r^2 dr$

$$\Rightarrow -\frac{4\pi}{\beta}\int_0^{+\infty} r^3 \frac{d}{dr}\left(e^{-\beta\varphi(r)} - 1\right) dr$$

$$= -\frac{4\pi}{\beta}\left\{\underbrace{r^3\left(e^{-\beta\varphi(r)} - 1\right)\Big|_0^{+\infty}}_{0} - \int_0^{+\infty} 3r^2\left(e^{-\beta\varphi} - 1\right) dr\right.$$

since if $\varphi(r) \sim \frac{1}{r^6}$ (LJ)
$\to r^3\left(e^{-\beta A/r^6} - 1\right) \underset{r\to\infty}{\sim} \frac{1}{r^3} \to 0$

$$= nk_BT(1 + b_2 n)$$

↳ 1st correction to ideal gas law

with $b_2 = \frac{1}{2}\int f(r)\, d\vec{r}$ 2nd virial coefficient

and $f(r) = e^{-\beta\varphi(r)} - 1$ the Mayer f-function
temp. dependent

@ low T: f has a strong peak
interpart. attraction dominates
$\Rightarrow b_2 < 0$, T_1
$\to P < P_{\text{ideal gas}}$

@ high T: attractive part inter. negligible
→ dominated by short dist. repulsion
$\Rightarrow b_2 > 0$, T_2
$\to P > P_{\text{ideal gas}}$

@ $T = T_{\text{Boyle}}$: $b_2 = 0$
→ behaviour v. close to ideal gas - corrections of higher order in n

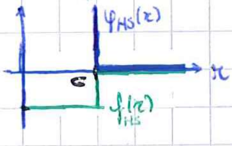
virial expansion:

$$P = nk_BT(1 + b_2 n + b_3 n^2 + \dots)$$

with b_k the k^{th} virial coeff
and $b_3 = -\frac{1}{3}\int d\vec{r}\, d\vec{r}'\, f(\vec{r}) f(\vec{r}') f(\vec{r} - \vec{r}')$

examples:

Hard spheres



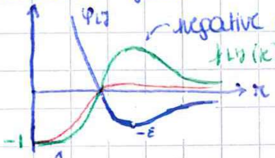
$$\varphi_{HS} = \begin{cases} 0 & r \geq \sigma \\ +\infty & r < \sigma \end{cases}$$

$$\to b_2 = -\frac{1}{2}\int_0^{+\infty} dr\, 4\pi r^2 \left(e^{-\beta\varphi_{HS}} - 1\right) = \frac{2\pi\sigma^3}{3} > 0 \text{, indep. on T}$$

$$= -2\pi\left[\int_0^{\sigma} r^2(-1)\,dr + \int_{\sigma}^{+\infty} dr\, r^2 \cdot 0\right.$$

→ hard spheres are athermal , $Z(N,V,T) = \frac{Q(N,V,\cancel{T})}{N!\,\lambda_T^{3N}}$

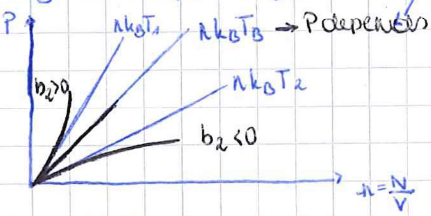
Lennard-Jones



if temp is high: $\beta\varphi \ll 1$, $\beta\epsilon \ll 1$
→ b_2 expected to be positive

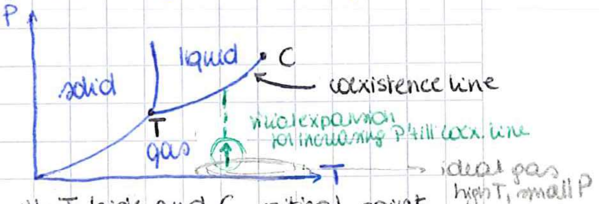
finding Boyle temp.

$P = nk_BT(1 + b_2 n + b_3 n^2 + \dots)$



↳ look @ the virial exponents to see how a gas deviates from the ideal gas

phase diagram real system



with T triple and C critical point

two phase can coexist if

$P_A = P_B$ & $T_A = T_B$ & $\mu_A = \mu_B$
mechanical equilibrium thermal equilibrium chemical equilibrium

The van der Waals model

$$P = \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2}$$ $a, b > 0$

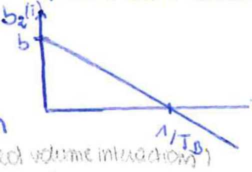
→ low density limit (n small)

$$P = \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2} = \frac{Nk_BT}{V}(1 + nb + n^2b^2 + \dots) - an^2$$

$$= nk_BT\left[1 + n\left(b - \frac{a}{k_BT}\right) + \dots\right]$$

compare with virial expansion and we find

$$b_2 = b - \frac{a}{k_BT}$$



$k_BT_B = \frac{a}{b}$ $[b_2(T_B) = 0]$

where thus

b should come from repulsion (excluded volume interaction)

a should come from attraction (intermolecular)

repulsion → $b_2 > 0$
attraction → $b_2 < 0$ ~

- Bogoliubov inequality

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ → interaction terms

$$\to \mathcal{Z} = \sum e^{-\beta\mathcal{H}} = \sum e^{-\beta\mathcal{H}_0} e^{-\beta\mathcal{H}_1} = \mathcal{Z}_0 \frac{\sum e^{-\beta\mathcal{H}_0} e^{-\beta\mathcal{H}_1}}{\underbrace{\sum e^{-\beta\mathcal{H}_0}}_{\mathcal{Z}_0}} = \mathcal{Z}_0 \langle e^{-\beta\mathcal{H}_1}\rangle_0$$

now $\langle e^{-\beta\mathcal{H}}\rangle_0 \geq e^{-\beta\langle\mathcal{H}\rangle_0}$

proof: for any $x \in \mathbb{R}$: $\exp(x) \geq 1 + x$

now x is a random variable distributed according to some distr. $p(x) \geq 0$

$\to \langle f\rangle = \int dx\, f(x)\, p(x)$

$\Rightarrow \langle e^{x - \langle x\rangle}\rangle \geq \langle 1 + (x - \langle x\rangle)\rangle = 1$

$\Rightarrow \langle e^x\rangle \geq e^{\langle x\rangle}$

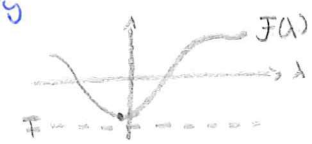
thus $\mathcal{Z} \geq \mathcal{Z}_0 e^{-\beta\langle\mathcal{H}_1\rangle_0}$

since $F = -k_BT \log\mathcal{Z}$ (Helmholtz free energy)

$\leq -k_BT\log\mathcal{Z}_0 + \langle\mathcal{H}_1\rangle_0$

$\Rightarrow$ $F \leq F_0 + \langle\mathcal{H}_1\rangle_0$ Bogoliubov inequality

free energy full system



if $\mathcal{H}(\lambda) = \mathcal{H}_0(\lambda) + \mathcal{H}_1(\lambda)$

$\to F \leq \mathcal{F}(\lambda) = F_0(\lambda) + \langle\mathcal{H}_1(\lambda)\rangle_0$

Bogoliubov : $\hat{F} = \min_\lambda \mathcal{F}(\lambda)$ is best approx of free energy

- Derivation van der Waals

$\mathcal{H} = \sum_{i=1}^N\left(\frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j}^N \varphi(q_{ij})\right)$ with $\varphi(r) = \begin{cases} +\infty & r < \sigma \\ -r^{-6} & r > \sigma \end{cases}$ → hard sphere pot + attractive tail ~ r^{-6}

$= \left(\sum_{i=1}^N \frac{\vec{p}_i^{\,2}}{2m} + N\lambda\right) + \left(\sum_{i=1}^N\sum_{i<j}^N \varphi(q_{ij}) - N\lambda\right) = \underbrace{\mathcal{H}_0(\lambda)}_{\text{ideal gas with a shifted constant potential}} + \mathcal{H}_1(\lambda)$

$\mathcal{Z}(\lambda) = \int \frac{d\Gamma}{N!\,h^{3N}} e^{-\beta\mathcal{H}_0(\lambda)}$

$\Rightarrow \mathcal{Z}_0 = \frac{V^N}{N!\,\lambda_T^{3N}} e^{-\beta N\lambda}$ $\to F_0(\lambda) = -Nk_BT\log\left(\frac{V}{N\lambda_T^3}\right) - Nk_BT + N\lambda$

→ $\log N! = N\log N - N$

$$\langle\mathcal{H}_1\rangle_0 = \frac{\int d\vec{q}_1\dots d\vec{q}_N\left(\sum_{i=1}^N\sum_{i<j}^N \varphi(q_{ij}) - N\lambda\right)e^{-\beta N\lambda}}{\int d\vec{q}_1\dots d\vec{q}_N\, e^{-\beta N\lambda}}$$

$\sum_{i=1}^N\sum_{i<j}^N \to \frac{N(N-1)}{2}$

$$= \frac{1}{V^N}\left\{\frac{N(N-1)}{2}\underbrace{\int d\vec{q}_1\dots d\vec{q}_N\, \varphi(q_{12})}_{V^{N-2}\int d\vec{q}_1 d\vec{q}_2\, \varphi(q_{12}) = V^{N-1}\int d\vec{q}\,\varphi(\vec{q})} - N\lambda V^N\right\}$$

$\vec{q} = \vec{q}_1 - \vec{q}_2$
$\vec{q}_{cm} = \frac{\vec{q}_1 + \vec{q}_2}{2}$

$$= \frac{N(N-1)}{2V}\int d\vec{q}\,\varphi(\vec{q}) - N\lambda \approx \frac{N^2}{2V}\int d\vec{q}\,\varphi(\vec{q}) - N\lambda$$

$\frac{N}{2}\int d\vec{q}\,\varphi(\vec{q})\, n$

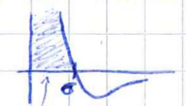
$$\to \mathcal{F}(\lambda) = F_0(\lambda) + \langle\mathcal{H}_1(\lambda)\rangle_0 = \underbrace{-Nk_BT\log\left(\frac{V}{N\lambda_T^3}\right) - Nk_BT}_{\text{ideal gas}} + \underbrace{\frac{N^2}{2V}\int d\vec{q}\,\varphi(\vec{q})}_{\text{corrections due to interactions}}$$

free energy approx. of interacting system

} taken into account with a spatial average here F independent of λ, no minimum necessary

$\rightarrow$ pressure $P = -\frac{\partial F}{\partial V} \approx -\frac{\partial \mathcal{F}}{\partial V} = \frac{Nk_BT}{V} + \frac{N^2}{2V^2}\int d\vec{q}\,\varphi(\vec{q})$

note how the integral diverges : $\varphi = \begin{cases} +\infty & r < \sigma \quad (1) \\ -1/r^6 & r > \sigma \quad (2) \end{cases}$



excluded volume
→ effective volume available: V-Nb →

⇒ split interaction pot. into 2 parts

(1) hard sphere repulsion → correction ideal gas law : $V \rightarrow V - Nb$

(2) attractive tail

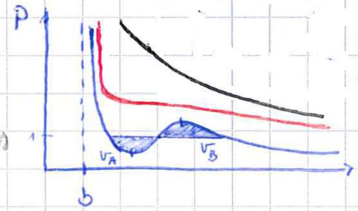
$\Rightarrow P \approx \frac{Nk_BT}{V-Nb} + \frac{N^2}{2V^2}\int_{r>\sigma} d\vec{r}\,\varphi(r)$

$= \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2}$ → van der Waals!

$a = -\frac{1}{2}\int_{|q|>\sigma} d\vec{q}\,\varphi(\vec{q})$

- Phase separation

below critical point
⇒ separation into 2 different phases

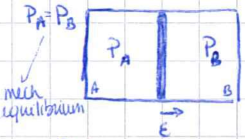


$T > T_c$ → $\frac{aN^2}{V^2}$ neglected, pressure decreases monotonically with V

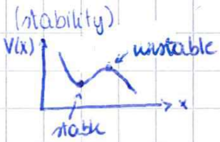
$T = T_c$

$T < T_c$ → for some values of V : $\frac{\partial P}{\partial V}\Big|_{N,T} > 0$ → thermodynamic instability

thermodyn. instability $\frac{\partial P}{\partial V} > 0$:

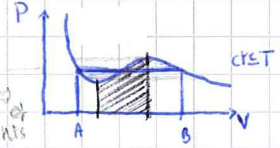


if $\frac{\partial P}{\partial V} < 0$: P_A decreases, P_B increases



if $\frac{\partial P}{\partial V} > 0$: unstable equilibrium

→ wall will move further until new stable equil. is reached



mech stability for each of these points $P_A = P_B$

→ phase separation : a dense (liquid) phase & a dilute (gas) phase

2 phase coexistence

OK! $\begin{cases} \mu_A = \mu_B \\ P_A = P_B = P_* \\ T_A = T_B \end{cases}$

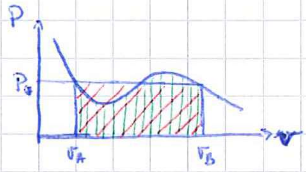
$\mu_A = \mu_B \rightarrow \mu N = E - TS + PV = F + PV = G$

thus $f_A + P_A v_A = f_B + P_B v_B$ $\qquad f = F/N$, $v = V/N$

$\Rightarrow f_A - f_B = P_*(v_B - v_A)$

$= -\int_{v_A}^{v_B} \frac{\partial f}{\partial v} dv$

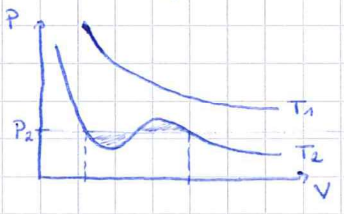
and since $P(v) = -\frac{\partial f}{\partial v}\Big|_{T,N} = \frac{\partial F}{\partial V}\Big|_{T,N}$



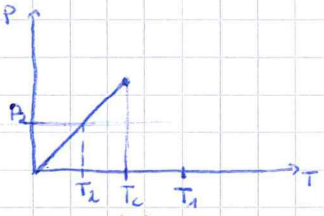
$\Rightarrow \int_{v_A}^{v_B} P(v)\,dv = P_*(v_B - v_A)$

Maxwell construction (equal area law)

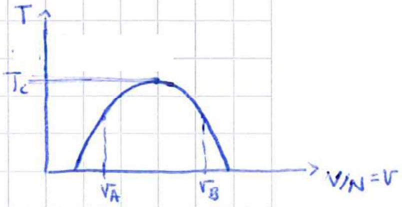
Some diagrams :)



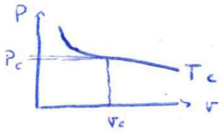
P(N,V,T)



critical point = endpoint phase coexistence



- Critical point and critical exponents



VDW $P = \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2}$ (1)

in vicinity critical point : $\frac{\partial P}{\partial V}\Big|_{N,T} = -\frac{Nk_BT}{(V-Nb)^2} + \frac{2aN^2}{V^3} = 0$ (2)

$\frac{\partial^2 P}{\partial V^2}\Big|_{N,T} = \frac{2Nk_BT}{(V-Nb)^3} - \frac{6aN^2}{V^4} = 0$ (3)

→ from these 3 eq , we find :

$v_c = \frac{V_c}{N} = 3b$, $P_c = \frac{a}{27b^2}$, $k_BT_c = \frac{8a}{27b}$ (*)

→ rescaled thermodynamic variables

$\tilde{p} = P/P_c$, $\tilde{T} = T/T_c$, $\tilde{v} = v/v_c$ $\xrightarrow[\text{VDW}]{}$ $\tilde{p}P_c = \frac{k_B\tilde{T}T_c}{\tilde{v}v_c - b} - a\frac{1}{\tilde{v}^2 v_c^2}$

⇒ VDW $\tilde{p} = \frac{8\tilde{T}}{3\tilde{v}-1} - \frac{3}{\tilde{v}^2}$ ↲ sub (*)

now in the vicinity of the critical point (@ critical point : $\tilde{p}=1, \tilde{T}=1, \tilde{v}=1$)

$\tilde{p} = 1+\delta p$, $\tilde{v} = 1+\delta v$ $\tilde{T} = 1+\delta t$ where $\delta p, \delta v, \delta t \ll 1$

→ VDW $(1+\delta p) = \frac{8(1+\delta t)}{3(1+\delta v)-1} - \frac{3}{(1+\delta v)^2} = \frac{8(1+\delta t)}{2+3\delta v} - \frac{3}{(1+\delta v)^2}$ ↲ expand

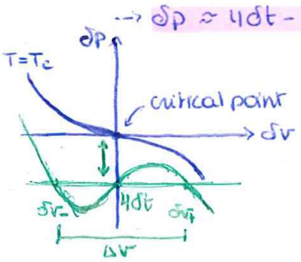
$\frac{1}{1-x} = (1+x+x^2+\dots)$

$\approx 8(1+\delta t)\frac{1}{2}\left(1 - \frac{3}{2}\delta v + \frac{9}{4}\delta v^2 - \frac{27}{8}\delta v^3\right) - 3(1 - 2\delta v + 3\delta v^2 - 4\delta v^3)$

$= 1 + 4\delta t - 6\delta t\delta v + 9\delta t(\delta v)^2 - \frac{3}{2}(\delta v)^3 + \dots$

$4\left(1 - \frac{3}{2}\delta v + \frac{9}{4}\delta v^2 - \frac{27}{8}\delta v^3 + \delta t - \frac{3}{2}\delta t\delta v + \frac{9}{4}\delta t\delta v^2 - \frac{27}{8}\delta t\delta v^3\right)$
$- 3 + 6\delta v - 9\delta v^2 + 12\delta v^3$
$= 4 + 4\delta t - 6\delta t\delta v + 9\delta t\delta v^2 - \left[\frac{27}{2} - \frac{24}{2}\right]\delta v^3 - 3$
$- 6\delta v + 6\delta v + 9\delta v^2 - 9\delta v^2 + O(\delta v^3) + O(\delta t^2)$

→ $\delta p \approx 4\delta t - 6\delta t\delta v + 9\delta t(\delta v)^2 - \frac{3}{2}(\delta v)^3$



1) for $T=T_c$ → $\delta t = 0$ along critical isotherm

$\delta p = -\frac{3}{2}(\delta v)^3$ → power law

2) for $T<T_c$ → $\delta t<0$ (approach T_c from below)

@ $\delta v = 0$: $\delta p = 4\delta t$

→ $4\delta t \approx 4\delta t - 6\delta t\delta v + 9\delta t(\delta v)^2 - \frac{3}{2}(\delta v)^3$

$D = 6^2\delta t^2 - 16\delta t$ ⇒ $\delta v^2 - 6\delta t\delta v + 4\delta t = 0$

→ $\frac{\sqrt{D}}{2} = \sqrt{\frac{36}{4}\delta t^2 - \frac{16}{4}\delta t}$ → $\delta v_\pm = 3\delta t \pm \sqrt{9\delta t^2 - 4\delta t} \underset{\delta t \to 0}{\approx} \pm\sqrt{-4\delta t} = \pm 2\sqrt{-\delta t}$

→ $\sqrt{\delta t}$ leading order

show the difference Δv between the volumes of the 2 coex. phases vanishes as δt→0

2) alternative method:
neglect $9\delta t(\delta v)^2$ since of higher order than $\delta t\delta v$ and $(\delta v)^3$
→ $\delta p = 4\delta t - 6\delta t\delta v - \frac{3}{2}(\delta v)^3$
now maxwell construction
$\delta v_A = -\delta v_B$
→ $\delta p(\delta v_A) = \delta p(-\delta v_A)$
⇒ $4\delta t - 6\delta t\delta v_A - \frac{3}{2}(\delta v_A)^3$
$= 4\delta t + 6\delta t\delta v_A + \frac{3}{2}(\delta v_A)^3$
⇒ $-12\delta t\delta v_A = 3(\delta v_A)^3$
→ $\delta v = \delta v_A - \delta v_B = 2\delta v_A \sim \sqrt{|\delta t|}$

maxwell construction $\Delta v = \delta v_+ - \delta v_- = 4\sqrt{-\delta t} = 4(-\delta t)^{1/2}$ → power law

→ 2 coexisting phases with $\delta v_A = -\delta v_B$

3) isothermal compressibility

$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_T$

now $\frac{\partial \tilde{p}}{\partial \tilde{v}} = \frac{-24(1+\delta t)}{(3\tilde{v}-1)^2} + \frac{6}{\tilde{v}^3} = -6(1+\delta t) + 6 = -6\delta t$

$= \frac{\partial P}{\partial v}\frac{v_c}{P_c} = \frac{v_c}{P_c}N\frac{\partial P}{\partial V} = \frac{V_c}{P_c}\frac{\partial P}{\partial V}$

⇒ $\kappa_T \sim \frac{1}{|\delta t|} = (\delta t)^{-1}$ → power

4) specific heat

$\lambda \sim \beta^{1/2}$

$E = -\partial_\beta \log Z = \frac{\partial(\beta F)}{\partial \beta} = \frac{3}{2}Nk_BT - \frac{aN^2}{V}$ $\beta F = -N\log\left(\frac{V}{N}\right) - N + N\log(\lambda_T^3) + \beta\frac{aN^2}{V}$

⇒ $c_V = \frac{\partial E}{\partial T}\Big|_V = \frac{3Nk_B}{2}$

→ $c_V \sim$ const $(\delta t)^0$ → power

$$\Rightarrow \begin{cases} \delta p \sim (\delta v)^\delta \\ \delta v \sim (\delta t)^\beta \\ \kappa_T \sim |\delta t|^{-\gamma} \\ c_V \sim |\delta t|^{-\alpha} \end{cases}$$

→

	α	β	δ	γ
VDW	0	1/2	3	1
fluid exp	0.13	0.33	4.8	1.24

} critical exponents (universal : do not dep. on a or b)

→ fluids @ liquid-vapor points (close to critical point)
→ performed in diff. fluids, all same value : universality

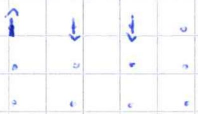
→ several quantities vanish or diverge as power laws in the vicinity of the critical point

Ising model

VDW fails to quantitatively predict the right exponents due to the used approx in the model – mean-field approx: interaction betw. part. accounted for in averaged way
↳ $\frac{N^2}{2V}\int d\vec{q}\,\varphi(\vec{q})$

⇒ model beyond mean field approx needed:
Ising model

lattice model
with a spin at
each lattice point



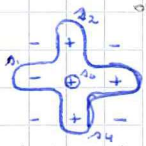
("magnetic model")

$s_i = \pm 1$ → for N spins : 2^N possible configurations

configuration:

$\mathcal{H} = -J \underbrace{\sum_{\langle ij \rangle}}_{\text{nearest-neighbours}} s_i s_j - H \sum_i s_i$

nearest-neighbours : → favors alignment of neighbouring spins



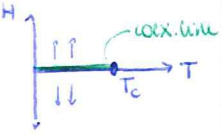
only coupling with nearest neighbours

with $J>0$ (ferromagn. case) the coupling strength between spins (interaction)
$H \gtrless 0$ the magnetic field (H=0 : minimal energy config, all spins "aligned")
↳ pos spin config have lower energies if H>0; neg spins if H<0
↳ spins want to follow H (lower energy state)

→ $Z = \sum_{\{s_i\}} e^{-\beta\mathcal{H}(\{s_i\})}$
↳ sum over 2^N states

→ average value of a spin: $\langle s_k \rangle = \frac{1}{Z}\sum_{\{s_i\}} e^{-\beta\mathcal{H}(\{s_i\})} s_k$

magnetization: $M = \Sigma_k \langle s_k \rangle$
↳ if $H \neq 0$ → spins predominantly aligned along direction H
→ $M \neq 0$



spontaneous magnetization : magn $M \neq$ in absence of magn field (H=0)
≡ $\lim_{H\to 0^\pm} m(T,H) = \pm m_0(T) \neq 0$ with $m = M/N$
⇒ phase transitions occur
@ coex. line : 2 diff. phases with opposite spontaneous magn

- Absence of phase transition in one dimension

1D :

$Z = \sum_{s_1=\pm1}\sum_{s_2=\pm1}\cdots\sum_{s_N=\pm1} e^{\beta(J\Sigma_{i=1}^N s_i s_{i+1} + H\Sigma_{i=1}^N s_i)}$

$= \mathrm{Tr}(T^N)$ → T is the transfer matrix with 2 eigenvalues $\lambda_\pm$: $T = \begin{pmatrix} e^{\beta J-\beta H} & e^{-\beta J-\beta H} \\ e^{-\beta J+\beta H} & e^{\beta J+\beta H} \end{pmatrix}$ → $\lambda^2 - 2e^{\beta J}\cosh(\beta H)\lambda + \sinh(\beta J) = 0$
→ $\lambda_\pm = e^{\beta J}\cosh(\beta H) \pm \sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(\beta J)}$

$= \lambda_+^N + \lambda_-^N \approx \lambda_+^N$ in limit $N\to\infty$, largest λ dominates

⇒ $F = -Nk_BT \log \lambda_+$

$\lambda_+ = e^{\beta J}\cosh(\beta H) + \sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(\beta J)}$

$\frac{\partial\log(\lambda_+)}{\partial H}\Big|_{H=0} = \frac{2\beta e^{2\beta J}\sinh(\beta H)}{\sqrt{e^{2\beta J}(2\cosh(2\beta H)+1) - 8\sinh(\beta J)}}\Big|_{H=0} = 0$

however no spontaneous magnetization at any finite temp
⇒ one-dim systems with short range interactions have no phase transitions at any finite temperature

$M = \Sigma_k \langle s_k \rangle$
$= \frac{\sum_{s_k}\sum_{\{s_i\}} e^{-\beta\mathcal{H}(\{s_i\})} s_k}{\underbrace{\sum_{\{s_i\}} e^{-\beta\mathcal{H}(\{s_i\})}}_{Z}}$
$= \frac{\partial \log Z}{\beta\partial H}$
$= -\frac{\partial(-k_BT\log Z)}{\partial H}$
$= -\frac{\partial F}{\partial H}$
↳ for spont. magn: $|_{H=0}$

however, at zero temp

TODO {

$\lim_{H\to\pm0}\lim_{T\to0^+} m(T,H) = \pm 1$ → spont. magn.

⇒ $T_c = 0$ for 1D Ising

- Two dimensional Ising model: exact solution

2D and 3D do have spont. magn
here is ferromagn

spontaneous magn for $T<T_c$:
$m_0(T) = [1 - \sinh^{-4}(\frac{2J}{k_BT})]^{1/8}$
with T_c derivable from
$\sinh(2J/k_BT) = 1$

in the vicinity of the critical point : $m_0(T) \sim (T_c - T)^{1/8}$
⇒ $\beta = 1/8$ since $m_0(T) \sim \delta v$ VDW

- Mean-field approximation

approximate solution Ising model using Bogoliubov

$$\rightarrow \mathcal{H} = \underbrace{\mathcal{H}_0(\lambda)}_{\text{non-int}} + \underbrace{\mathcal{H}_1(\lambda)}_{\text{interacting}}$$

$$= -\lambda \sum_i s_i - \mathcal{J} \sum_{\langle ij \rangle} s_i s_j - (H-\lambda) \sum_i s_i$$

$$\mathcal{Z}_0 = \sum_{\{s_i\}} e^{+\beta\lambda \sum_i s_i} = \sum_{s_1=\pm1} \sum_{s_2=\pm1} \dots \sum_{s_N=\pm1} e^{\beta\lambda s_1} e^{\beta\lambda s_2} \dots e^{\beta\lambda s_N}$$

$$= \Big(\sum_{s=\pm1} e^{\beta\lambda s}\Big)^N = \big(e^{\beta\lambda} + e^{-\beta\lambda}\big)^N = (2\cosh\beta\lambda)^N$$

$$\rightarrow F_0 = -k_B T \log \mathcal{Z}_0 = -N k_B T \log(2\cosh\beta\lambda)$$

$$\langle \mathcal{H}_1(\lambda) \rangle_0 = \frac{1}{\mathcal{Z}_0} \sum_{\{s_i\}} \Big\{ -\mathcal{J} \sum_{\langle ij \rangle} s_i s_j - (H-\lambda) \sum_i s_i \Big\} e^{-\beta \mathcal{H}_0}$$

they do not interact in $\mathcal{H}_0$

$$= -\mathcal{J} \sum_{ij} \langle s_i s_j \rangle_0 - (H-\lambda) \sum_i \langle s_i \rangle_0 = -\mathcal{J} \frac{Nz}{2} \langle s \rangle_0^2 - (H-\lambda) N \langle s \rangle_0$$

z = # neighbours each spin has = 2 · dimension
N spins
do not consider each pair twice ⇒ $\frac{1}{2}$

$$\rightarrow \mathcal{F}(\lambda) = F_0(\lambda) + \langle \mathcal{H}_1(\lambda) \rangle_0$$

$$= -N k_B T \log(2\cosh\beta\lambda) - \frac{\mathcal{J} z N}{2} \langle s \rangle_0^2 - (H-\lambda) N \langle s_* \rangle_0$$

$$= -N k_B T \log(2\cosh\beta\lambda) - \frac{\mathcal{J} z N}{2} \tanh^2(\beta\lambda) - (H-\lambda) N \tanh(\beta\lambda)$$

$$\langle s \rangle_0 = \frac{1}{N} \frac{\partial \log \mathcal{Z}}{\beta \partial \lambda} = \frac{\partial \log(\cosh\beta\lambda)}{\beta \partial \lambda} = \tanh(\beta\lambda)$$

$$\rightarrow \tilde{F} = \lim_{\lambda} \mathcal{F}(\lambda)$$

$$\frac{\partial}{\partial \lambda} \qquad \frac{\partial}{\partial \langle s \rangle_0}\Big|_\lambda \frac{\partial \langle s \rangle_0}{\partial \lambda}$$

$$\frac{\partial \mathcal{F}}{\partial \lambda}\Big|_{\lambda=\lambda_*} = 0 \Rightarrow \underbrace{-N \tanh\beta\lambda_* + N \langle s \rangle}_{0} - N \frac{\partial \langle s \rangle_0}{\partial \lambda} \big(\mathcal{J} z \langle s \rangle_0 + H - \lambda\big)\Big|_{\lambda_*} = 0$$

$$\Rightarrow N \frac{\partial \langle s \rangle_0}{\partial \lambda} \big(\mathcal{J} z \langle s \rangle_0 + H - \lambda_*\big) = 0 \Rightarrow \mathcal{J} z \langle s \rangle_0 + H - \lambda_* = 0$$

$$\Rightarrow \lambda_* = H + \mathcal{J} z \tanh(\beta\lambda_*)$$

$$\tilde{F} = \mathcal{F}(\lambda_*)$$

$$= -N k_B T \log(2\cosh\lambda_*) - \frac{N\mathcal{J}z}{2} \tanh^2(\beta\lambda_*) - N(H-\lambda_*) \tanh(\beta\lambda_*)$$

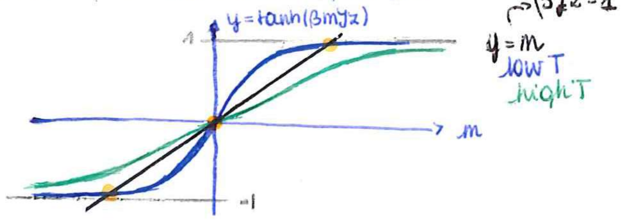
$$m = -\frac{\partial \tilde{F}}{N \partial H} = \frac{-1}{N} \Big[\underbrace{\frac{\partial \tilde{F}}{\partial \lambda_*} \frac{\partial \lambda_*}{\partial H}}_{0 \text{ since } \lambda_* \text{ minimizes } \tilde{F}} + \frac{\partial \tilde{F}}{\partial H}\Big|_\lambda \Big]_T$$

$$= \tanh(\beta\lambda_*) = \tanh[\beta(m\mathcal{J}z + H)]$$

$m = \tanh(\beta\lambda_*)$
$\Rightarrow \tanh(\beta[H + \mathcal{J}z \tanh(\beta\lambda_*)])$
$= \tanh(\beta[H + \mathcal{J}zm])$

now spontaneous magnetization?

$H = 0 \rightarrow m = \tanh(\beta m \mathcal{J} z)$



at "high" T, low β ($\beta\mathcal{J}z < 1$)
only 1 solution: $m = 0$

at sufficiently low T, high β ($\beta\mathcal{J}z > 1$)
spont. magn is possible:
$m = \tanh(\beta m \mathcal{J} z)$ also has
2 additional solutions: $m = \pm m_0 \neq 0$

critical point

$$\frac{d \tanh(\beta m \mathcal{J} z)}{dm}\Big|_{m=0} = 1$$

$$\tanh(\beta m \mathcal{J} z) \approx \beta m \mathcal{J} z - \frac{1}{3}(\beta m \mathcal{J} z)^3$$

$$\beta_c \mathcal{J} z - (\beta_c m \mathcal{J} z)^2\Big|_{m=0} = \beta_c \mathcal{J} z = 1$$

$$\Rightarrow k_B T_c = \mathcal{J} z$$

approximation becomes better at higher dimensions
we've approx. the effect of spin-spin interactions with an average field λ
a given spin s_j appears in the Hamiltonian with a term of type

$$\big(\textstyle\sum_i' s_i + H\big) s_j$$

where the sum is over all its neighbours
if each spin has many neighbours, the approx $\sum_i' s_n \approx$ cte becomes more and more accurate

Critical exponents

several quantities vanish with power-laws in the vicinity of the critical point
→ critical exponents

spontaneous magn
$m_0(T) \sim (T_c - T)^\beta$ along coex line $H=0$
$c \sim |T - T_c|^\alpha$
$\chi = \frac{\partial M}{\partial H} \sim |T - T_c|^{-\gamma}$
↳ magn susceptibility
$H \sim |M|^d$ @ $T = T_c$

	α	β	γ	δ
Ising 2d	0	1/8	7/4	15
3d	0.13	0.33	1.24	4,8
Bragg-Williams mean field	0	1/2	1	3

→ identical to experim values for fluids

→ $m = \tanh[\ldots]$
↳ same as mean field VDW
↙ BMF exponents do not dep on dim
dim ~ z

correspondance VDW ⟷ Ising model: P ⟷ H
V ⟷ M

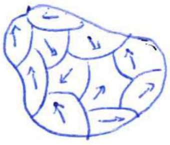
- $H \sim m^3$ @ $T = T_c$
  $k_B T_c = Jz$ → $m = \tanh(m + \beta_c H)$
  $\Rightarrow \cancel{m} \approx \cancel{m} + \beta_c H - \frac{1}{3}(m + \beta_c H)^3$
  $\Rightarrow m^3 \sim H$
- correlation length ξ = char. distance at which two spins are correlated
  = measure of the char. length scale of fluctuations of the syst.
  @ $T = T_c$: $\xi \sim |T_c - T|^{-\nu}$ → no longer a char lengthscale,
  system is scale invariant
  fluctuations occur at all length scales
  $G^{(2)} \sim \frac{1}{x^{d-2+\eta}}$ the correlation function

correlation function $G^{(2)}$
for $T > T_c$, $H = 0$ (absence spont. magn):
$G^{(2)}(\vec{x}, \vec{y}) = \langle s_{\vec{x}} s_{\vec{y}} \rangle$
spins close by → aligned (more likely)
far away spins → weakly interacting → more likely uncorrelated
for $T < T_c$: $m_0 \neq 0$, $\langle s \rangle \neq 0$
$G^{(2)}(\vec{x}, \vec{y}) = \langle (s_{\vec{x}} - \langle s \rangle)(s_{\vec{y}} - \langle s \rangle) \rangle$ (subtr. spin average)
for large distances both decay exponentially
$G^{(2)}(\vec{x}, \vec{y}) \sim e^{-\frac{|\vec{x} - \vec{y}|}{\xi}}$

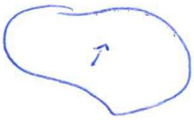
critical opalescence in fluids:
in vicinity of critical points fluctuations become of a size comparable to the wavelength of light → light is scattered
⇒ normally transparent liquid appears cloudy

↳ correlation length ξ diverges at critical point
→ critical behavior indep on local properties of the Hamiltonian
only determined by global properties (dim & symm)
⇒ many different systems share the same critical behaviour

Magnetism

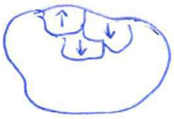


paramagnet   all points magnetized



ferromagnet

all points same magnetization



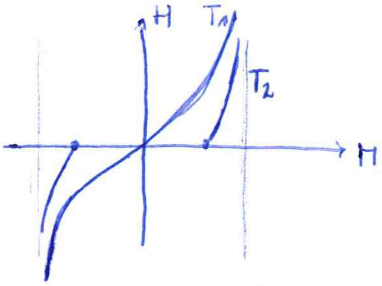
uniaxial ferromagnet

with thermodynamical variables

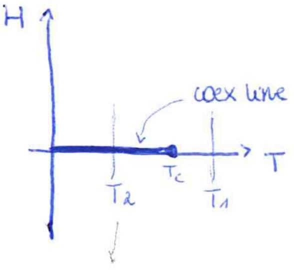
$M = \Sigma_i \mu_i$ magnetization (μ_i = magn. moments)

H magnetic field (external)

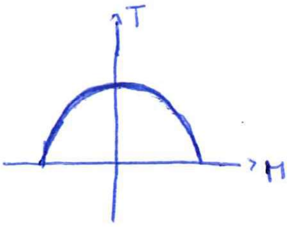
T temperature



at very large H magnetization → asymptotes



coexistence between 2 phases



[4] Quantum Statistical Mechanics

Before: $\rho(\Gamma) = \rho(P,Q)$ the prob. distribution for 6N-dim config.

NOW: + Heisenberg's uncertainty principle

$\Delta p_x \Delta x \geq \hbar/2$ (no simultanious info about mom. & pos.)

→ $\rho(P,Q)$ (prob. distr.) can't be

cannot specify state with Γ

+ Built-in probabilities

QM state given by $|\Psi_\alpha\rangle$; 1 part. w mass M ~ $\Psi(\vec{r},t)$

→ SE: $i\hbar\,\partial_t \Psi(\vec{r},t) = \hat{H}\Psi(\vec{r},t)$

with $\hat{H} = -\hbar^2/2\mu\, \vec{\nabla}^2 + V(\vec{r},t)$

↳ $|\Psi_\alpha\rangle$ eigenstates $\hat{H}$ with energy E_α

→ $|\Psi(\vec{q},\tau)|^2$: prob. to find particle at time τ, position $\vec{q}$

⇒ QM states & probabilities for system in state $|\Psi_\alpha\rangle$

* QM probability

for operator $\hat{A}$ (physical observable):

$$\langle\hat{A}\rangle_{pure} = \int \Psi_\alpha^*(q)\,\hat{A}\,\Psi_\alpha(q)\,d^{3N}q = \langle\Psi_\alpha|\hat{A}|\Psi_\alpha\rangle$$

+ Quantum statist. mech. prob

$$\langle\hat{A}\rangle = \sum_\alpha c_\alpha \langle\Psi_\alpha|\hat{A}|\Psi_\alpha\rangle_{pure}$$

with c_α the prob to find the syst. in state $|\Psi_\alpha\rangle$

norm.: $\sum_\alpha c_\alpha = 1$

$\langle\hat{A}\rangle = \sum_\alpha c_\alpha \sum_n \langle\Psi_\alpha|n\rangle\langle n|\hat{A}|\Psi_\alpha\rangle$

$= \sum_n \langle n|\hat{A} \left(\sum_\alpha c_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|\right)|n\rangle$

$= \sum_n \langle n|\hat{A}\hat{\rho}|n\rangle$

Using $\sum_n |n\rangle\langle n| = \mathbb{1}$ [complete set of orthonorm. states]

⇒ $\langle\hat{A}\rangle = Tr_{\mathcal{H}}(\hat{A}\hat{\rho})$ where $\hat{\rho} \equiv \sum_\alpha c_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|$

$= \sum_n \langle n|\hat{A}\hat{\rho}|n\rangle$ the density matrix

classical equivalent $\langle A\rangle = \int d\Gamma \rho(\Gamma) A(\Gamma)$

classical equivalent prob. distr. function $\rho(\Gamma)$

special case: pure state := $\rho = |\phi\rangle\langle\phi|$ (no summ)

→ Quantum canonical ensemble

since $\hat{H}|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle$

and $c_\alpha = 1/Z\; e^{-\beta E_\alpha}$ specifies prob distr of states

with $\beta = 1/k_BT$

$Z = \sum_\alpha e^{-\beta E_\alpha}$ [norm factor] = PARTITION FUNCTION

→ $\hat{\rho}_{can} = \exp(-\beta\hat{H}) / Tr(\exp(-\beta\hat{H}))$

↳ $f(M) = \begin{pmatrix} f(M_{11}) & & \\ & f(M_{22}) & \\ & & \ddots \; f(M_{nn}) \end{pmatrix}$ for M diag.

⇒ $e^{\hat{M}} = \mathbb{1} + \hat{M} + \frac{1}{2}\hat{M}\hat{M} + \frac{1}{3!}\hat{M}\cdot\hat{M}\cdot\hat{M} + \ldots$

example: Quantum Harm. Osc.

$$\hat{H} = -\hbar^2/2\,\partial_x + \omega^2/2\; x^2$$

→ $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1$

with hermite polynomials as Ψ_α, and ω the frequency

⇒ $E_n = \hbar\omega(n + 1/2)$ $n \in \mathbb{N}$ $\{0,1,2,\ldots\}$ (non-degenerate energy spectr.)

→ $Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{2\sinh(\beta\hbar\omega/2)}$

$\sum_{n=0}^{\infty} e^{-\beta E_n}$

$\sum_n e^{-\beta\hbar\omega(n+1/2)}$

$= e^{-\beta\hbar\omega/2} \sum_n e^{-\beta\hbar\omega n}$

$e^{-\beta\hbar\omega/2} \frac{1}{1-e^{-\beta\hbar\omega}}$

using $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

→ $\langle E\rangle = -\partial_\beta \log Z = \partial_\beta\left[\frac{\beta\hbar\omega}{2} + \log(1 - e^{-\beta\hbar\omega})\right]$

$= \hbar\omega\left[\frac{1}{2} + \underbrace{1/e^{\beta\hbar\omega} - 1}_{\langle n\rangle}\right]$

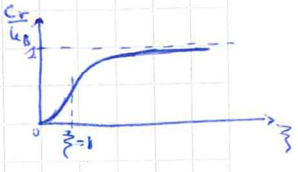
dimensionful $c_v = \partial E/\partial T = k_B(\hbar\omega/k_BT)^2\; e^{-\beta\hbar\omega}/(1-e^{-\beta\hbar\omega})^2$

$\xi = k_BT/\hbar\omega$

dimensionless → $\frac{c_v}{k_B} = \frac{1}{\xi^2}\cdot\frac{e^{-1/\xi}}{(1-e^{-1/\xi})^2}$

↱ equip. $1 - 1/12\,\xi^2$ $\xi \gg 1$ high T

↳ 0 purely quantum effect which peaks at some point $\xi \ll 1$ low T



⇒ $\xi \approx 1$ → $T_* = \hbar\omega/k_B$ crossover from QM → CM

↳ charact. temp

$T \gg T_*$ CM

$T \ll T_*$ QM

Fermions & Bosons

↪ symm under the exchange of coord. of any pair of part.

QM: particles indistinguishable $\Rightarrow |\Psi(\vec{q}_1,\dots,\vec{q}_i,\dots,\vec{q}_j,\dots,\vec{q}_N)|^2 = |\Psi(\vec{q}_1,\dots,\vec{q}_j,\dots,\vec{q}_i,\dots,\vec{q}_N)|^2$ for $i \neq j$

→ In $D \neq 2+1$ (2pos + 1 time) 2 options

1) Ψ is symmetric $i \leftrightarrow j$ → bosons: $s = 0, 1, 2, \dots$ (integer spin) (phase 0)

2) Ψ is antisymmetric $i \leftrightarrow j$ → fermions: $s = 1/2, 3/2, \dots$ (half-integer spin) (phase 1/2)

$D=2+1$ → anyons

example 2 non-interacting particles

- $\mathcal{H} = \mathcal{H}_1(\vec{p}_1,\vec{q}_1) + \mathcal{H}_1(\vec{p}_2,\vec{q}_2)$

  with $\mathcal{H}_1 = -\hbar^2/2m\,\vec{\nabla}_1^2 + V(\vec{q}_1)$ and $\hat{\mathcal{H}}_1|\phi_k\rangle = \varepsilon_k|\phi_k\rangle$ w k a quantum number

- quantum state:

  $\Psi(\vec{q}_1,\vec{q}_2) = \phi_k(\vec{q}_1)\phi_l(\vec{q}_2)$ with $k \neq l$

→ $\mathcal{H}\Psi = [\mathcal{H}_1(\vec{p}_1,\vec{q}_1) + \mathcal{H}_2(\vec{p}_2,\vec{q}_2)]\phi_k(\vec{q}_1)\phi_l(\vec{q}_2) = (\mathcal{H}_1(\vec{p}_1,\vec{q}_1)\phi_k(\vec{q}_1))\phi_l(\vec{q}_2) + \phi_k(\vec{q}_1)(\mathcal{H}_2(\vec{p}_2,\vec{q}_2)\phi_l(\vec{q}_2))$

$= (\varepsilon_k + \varepsilon_l)\Psi(\vec{q}_1,\vec{q}_2)$

However: no symm: $|\Psi(\vec{q}_1,\vec{q}_2)|^2 \neq |\Psi(\vec{q}_2,\vec{q}_1)|^2$ $k \neq l$

$\Psi(\vec{q}_2,\vec{q}_1) = \phi_k(\vec{q}_2)\phi_l(\vec{q}_1)$
$+ \phi_k(\vec{q}_1)\phi_l(\vec{q}_2)$

⇒ (anti)symmetrization

$$\Psi_\pm(\vec{q}_1,\vec{q}_2) = \frac{1}{\sqrt{2}}\left[\phi_k(\vec{q}_1)\phi_l(\vec{q}_2) \pm \phi_k(\vec{q}_2)\phi_l(\vec{q}_1)\right]$$

→ Ψ_+ : wavefunction for 2 ident. BOSONS (photons, phonons)

Ψ_- : " " 2 ident. FERMIONS (electrons, protons)

NOW: 2 particles in same quantum states

$\Psi(\vec{q}_1,\vec{q}_2) = \phi_k(\vec{q}_1)\phi_k(\vec{q}_2)$

→ $\Psi_+ = \sqrt{2}[\phi_k(\vec{q}_1)\phi_k(\vec{q}_2)]$ → Bosons ok

$\Psi_- = 0$ → Fermions: PAULI EXCLUSION PRINCIPLE "2 identical fermions cannot occupy the same quantum state"

N identical particles

n_γ = occupation # (# of particles) in state with ε_γ

↪ for example above (N=2): $n_k = n_l = 1$, $n_j = 0$ for $j \neq k, l$

general: * bosons: $n_\gamma \geq 0$

* fermions: $n_\gamma = 0, 1$ (Pauli)

→ $N = \sum_\gamma n_\gamma$, $E = \sum_\gamma n_\gamma \varepsilon_\gamma$

tot amount particles — total energy

→ canonical ensemble: partition function

$$Z(N,V,T) = \sum_{n_1}\sum_{n_2}\dots\sum_{n_\gamma}\dots e^{-\beta\sum n_\gamma\varepsilon_\gamma}\underbrace{\delta_{N,\sum_\gamma n_\gamma}}_{\text{restricts \# part to N}}$$

makes expr. complicated

$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

→ grand canonical ensemble: part. function

change to

$$\Xi(\mu,V,T) = \sum_{N=0}^{\infty} Z(N,V,T)\,e^{\beta\mu N} = \sum_{\{n_\gamma\}} e^{-\beta\sum_\gamma n_\gamma\varepsilon_\gamma}\underbrace{\sum_{N=0}^{\infty} e^{\beta\mu N}\delta_{N,\sum_\gamma n_\gamma}}$$

$\sum_{\{n_\gamma\}} = \sum_{n_1}\sum_{n_2}\dots$

$$= \sum_{\{n_\gamma\}} e^{-\beta\sum_\gamma n_\gamma\varepsilon_\gamma}\,e^{\beta\mu\sum_\gamma n_\gamma}$$

$$= \prod_\gamma \sum_{n_\gamma} e^{-\beta n_\gamma(\varepsilon_\gamma - \mu)}$$

$e^{a+b} = e^a e^b$

$$\Xi(\mu,V,T) = \prod_\gamma \sum_{n_\gamma} e^{-\beta n_\gamma(\varepsilon_\gamma-\mu)}$$

→ Bosons

! convergent only for $e^{\beta(\mu-\varepsilon_\gamma)} < 1 \rightarrow \mu < \min(\varepsilon_\gamma)$ upperbound μ

+ uses geom. series $\left(\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}\right)$

$$\Rightarrow \Xi_{BE} = \prod_\gamma \frac{1}{1-e^{\beta(\mu-\varepsilon_\gamma)}}$$

→ Fermions

$$\Xi_{FD} = \prod_\gamma \left[1 + e^{\beta(\mu-\varepsilon_\gamma)}\right]$$

$(n_\gamma = 0,1)$ no cond. on μ

$$\Rightarrow \log \Xi = \mp \sum_\gamma \log\left(1 \mp e^{\beta(\mu-\varepsilon_\gamma)}\right)$$

upper sign = bosons
lower sign = fermions

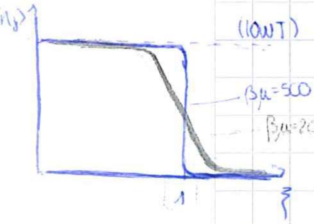
$$\rightarrow \langle n_\gamma\rangle = {-\partial \log \Xi}/{\beta\partial\varepsilon_\gamma} = \frac{1}{(e^{\beta(\varepsilon_\gamma-\mu)} \mp 1)}$$

↳ $\exp(\beta(\varepsilon_\gamma-\mu)) \geq 1$ bosons → $\langle n_\gamma\rangle > 0$

> 0 fermions → $\langle n_\gamma\rangle \leq 1$

$$E = {-\partial \log \Xi}/{\partial\beta}\Big|_{\beta\mu = cts} = \sum_\gamma \frac{\varepsilon_\gamma}{e^{\beta(\varepsilon_\gamma-\mu)} \mp 1} = \sum_\gamma \langle n_\gamma\rangle \varepsilon_\gamma$$

↳ Comments



bosons good approx
photons & phonons
fermions

* FD: $\langle n_\gamma\rangle_{FD} = \frac{1}{\exp(\beta\mu(\xi-1))+1}$ $\xi \equiv \varepsilon_\gamma/\mu$ (dimless)

→ value to which all energy states are occupied

low T $(\beta\to\infty)$: $\langle n_\gamma\rangle = 1$ for $\varepsilon_\gamma < \mu$ $(\xi < 1$ → occupied)

~ step function $\langle n_\gamma\rangle = 0$ for $\varepsilon_\gamma > \mu$ $(\xi > 1$ → free)

→ zero temperature limit: $\lim_{T\to 0} \mu(T) = \varepsilon_F$ fermi-energy

* BEC: $\langle n_\gamma\rangle_{BE} \gg 1$ $(\to\infty)$

$\Rightarrow \mu \to \varepsilon_\gamma$

* Bosons with $\mu = 0$: $\langle n_\gamma\rangle = 1/e^{\beta\varepsilon_\gamma - 1}$ – same as QHO

Quantum corrections to ideal gas law

(particles in a box)

system of non-interacting quantum particles in a cubic box of size L with single particle energy levels

$$\varepsilon_p = \vec{p}^2/2m$$

SE → $-\hbar^2/2m \nabla^2\phi_p = \varepsilon_p\phi_p$ $\Rightarrow \phi_{\vec{k}} = e^{i\vec{k}\cdot\vec{x}}$ (plane wave) /w ~~[illegible]~~ $\vec{p} = \hbar\vec{k}$

impose periodic boundary cond → quantization $\vec{p}$?

$e^{ik_x(x+L)} = e^{ik_x x}$ $\Rightarrow k_x = \frac{2\pi}{L} n_x$ $\Rightarrow \vec{p} = \frac{h}{L}(n_x, n_y, n_z)$ $\in \mathbb{Z}$

controls spacing between energy levels

now: $\log \Xi = \mp \sum_{\vec{p}} \log\left(1 \mp e^{\beta(\mu - \vec{p}^2/2m)}\right)$

$= \mp \frac{V}{h^3} \int d\vec{p} \log\left(1 \mp \exp(\beta(\mu - \vec{p}^2/2m))\right)$

↙ $\int_{x_a}^{x_b} f(x)dx = \sum_i f(x_i)\Delta x_i$ small & $\Delta p = h/L$ & $L^3 = V$

$z = e^{\beta\mu}$ ↳

↙ $\log(1 \mp x) = -\sum_{l=1}^{\infty} (\pm 1)^l \frac{x^l}{l}$

$= \pm \frac{V}{h^3} \int d\vec{p} \left[\sum_{l=1}^{\infty} (\pm 1)^l \frac{z^l}{l} e^{-\beta\vec{p}^2 l/2m}\right]$

$4\pi \int dp\, p^2 e^{-\beta\vec{p}^2 l/2m} = 4\pi \left(\frac{1}{\beta l/2m}\right)^{3/2} \sqrt{\pi}/4$

bosons:
$\mu < \varepsilon_0 = 0$
$\beta > 0$
→ $z < 1$

$= \pm \frac{V}{h^3} \pi^{3/2} (2mk_BT)^{3/2} \sum_{l=1}^{\infty} (\pm 1)^l z^l / l^{5/2}$

↙ $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$

$= \pm \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} (\pm 1)^l z^l / l^{5/2}$

$$\log \Xi = \pm \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} (\pm 1)^l \frac{z^l}{l^{5/2}}$$

$z = e^{\beta\mu}$

$$\rightarrow N = \frac{1}{\beta}\partial_\mu \log \Xi = \pm \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} (\pm 1)^l \frac{z^l}{l^{3/2}}$$

$$\Rightarrow \begin{cases} \frac{PV}{k_BT} = \pm \frac{V}{\lambda_T^3} \sum_{l=1}^{\infty} (\pm 1)^l \frac{z^l}{l^{5/2}} \\ n\lambda_T^3 = \pm \sum_{l=1}^{\infty} (\pm 1)^l \frac{z^l}{l^{3/2}} \end{cases}$$

$n = \frac{N}{V}$

$\log \Xi = \frac{PV}{k_BT}$

→ ~~lowest order~~ expand in z (small)

1st order $\begin{cases} \frac{PV}{k_BT} = \frac{V}{\lambda_T^3} z \\ n\lambda_T^3 = z \end{cases}$ $\Rightarrow p = nk_BT$ classical ideal gas

* using

$$\frac{x + \alpha x^2}{x + \beta x^2} \approx \frac{1+\alpha x}{1+\beta x}$$

$$\approx (1+\alpha x)(1-\beta x)$$

$$\hookrightarrow (1 \pm z/4\sqrt{2}) (1 \mp z/2\sqrt{2})$$

$$\approx 1 \pm z/4\sqrt{2} \mp z/2\sqrt{2} \mp O(z^2)$$

$$\approx 1 \pm z/4\sqrt{2} \mp 2z/4\sqrt{2}$$

2nd order $\begin{cases} \frac{PV}{k_BT} \approx \frac{V}{\lambda_T^3}\left(z \pm \frac{z^2}{4\sqrt{2}}\right) \\ n\lambda_T^3 \approx z \pm \frac{z^2}{2\sqrt{2}} \end{cases}$

$$p = nk_BT \frac{(z \pm z^2/4\sqrt{2})}{(z \pm z^2/2\sqrt{2})} \overset{*}{\underset{z \ll 1}{\approx}} nk_BT\left(1 \mp \frac{z}{4\sqrt{2}}\right)$$

$$\Rightarrow p \approx \underbrace{nk_BT}_{\text{classical}} \left(1 \mp \underbrace{\frac{n\lambda_T^3}{4\sqrt{2}}}_{\text{quantum correction}}\right)$$

3rd order $\begin{cases} \frac{PV}{k_BT} = \frac{V}{\lambda_T^3}\left(z \pm \frac{z^2}{4\sqrt{2}} + \frac{z^3}{9\sqrt{3}}\right) \\ n\lambda_T^3 = z \pm \frac{z^2}{2\sqrt{2}} + \frac{z^3}{3\sqrt{3}} \end{cases}$

$$p = nk_BT\left[1 \mp \frac{z}{4\sqrt{2}} + \left(\frac{1}{16} - \frac{2}{9\sqrt{3}}\right)z^2\right]$$

$$\Rightarrow p \approx nk_BT\left(1 \mp \frac{n\lambda_T^3}{4\sqrt{2}} + \left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)n^2\lambda_T^6\right)$$

Bose-Einstein condensation

For (free) bosons we considered

$$n\lambda_T^3 = \sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}}$$ > inverting

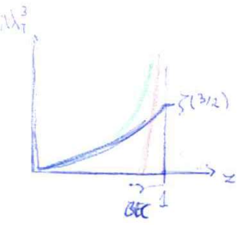
with $z = e^{\beta\mu}$, $n = N/V$, from $\Xi = \prod_r \frac{1}{1 - \exp(\beta(\mu - \epsilon_r))}$

↳ $z < 1$ since $\beta > 0$, $\mu < \epsilon_0 = 0$
→ else series diverges (for $z > 1$)

Now $\mu = \beta^{-1}\log z$ → $\mu(N,V,T)$

Then $P(N,V,T)$ by replacing $z = e^{\beta\mu}$ by pressure since $\frac{PV}{k_BT} = \frac{V}{\lambda_T^3}\sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}}$

→ $n\lambda_T^3 = \sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}}$ converges for $z \leq 1$, diverges for $z > 1$



for $z \to 1$, $\mu \to \epsilon_0$: $n\lambda_T^3 = \sum_{l=1}^{\infty} \frac{1}{l^{3/2}} = \zeta(3/2) \approx 2{,}612\ldots$

with $\zeta(s)$ Riemann zeta
$\zeta(s) \equiv \sum_{l=1}^{\infty} \frac{1}{l^s}$
$\zeta = 0$ for $s = -2n$ or $s = \frac{1}{2} + iy$
$n \in \mathbb{N}_0$ $y \in \mathbb{R}$
↳ Riemann Hypothesis

⇒ can only find a chem. potential for $n\lambda_T^3 \leq 2{,}612$
else unphysical result (no inversion possible)
(no μ)

→ reevaluate analysis:

$$N = \sum_{\vec{p}} \langle n_{\vec{p}} \rangle = \sum_{\vec{p}} \frac{1}{e^{\beta(p^2/2m - \mu)} - 1} = \underbrace{\frac{z}{1-z}}_{\substack{p=0 \\ \text{or ground} \\ \text{state contribution}}} + \underbrace{\sum_{\vec{p} \neq 0} \frac{z}{e^{\beta p^2/2m} - z}}_{\substack{p \neq 0 \\ \text{excited states} \\ \text{contribution}}}$$

for $z \to 1^-$ the 1st term diverges
this isn't taken into account in the integral form (z.o.z.)

$$N = \frac{z}{1-z} + \frac{4\pi V}{h^3}\sum_{l=1}^{\infty} z^l \int dp\, p^2 e^{-l\beta p^2/2m}$$

$$\frac{\sqrt{\pi}}{4}\left(\frac{2m}{l\beta}\right)^{3/2}$$

$$N = \frac{z}{1-z} + \frac{V}{\lambda_T^3}\sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}}$$

"separating ground state contr from excited states" ↳

$$n\lambda_T^3 = \frac{\lambda_T^3}{V}\frac{z}{1-z} + \sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}}$$

* low n, high T ($n\lambda_T^3 < 2{,}612$):
"classical regime", can neglect 1st term ~ $O(1/V)$

* high n, low T ($n\lambda_T^3 > 2{,}612$):
take into account ground state ~ $O(1)$
since a macroscopic number of particles can occupy the ground state
→ BEC

correction term for ground state contributions not taken into account for fermions
since only 1 fermion can occupy that state → minimal mistake

estimate

$n \lambda_T^3 \approx \zeta(3/2) \Rightarrow (N/V)^{2/3}\ h^2/2\pi m k_B T \approx (\zeta(3/2))^{2/3}$

$\Rightarrow k_B T_{BEC} \approx \frac{h^2}{2\pi m}\left(\frac{N}{V\zeta(3/2)}\right)^{2/3}$

→ atoms/mol. superfluids at low T

general (also valid for non-interacting bosons subject to external pot.)

bosons w density of states $g(\epsilon)$

$[\, g(\epsilon)d\epsilon = \#$ of states w energy $[\epsilon, \epsilon+d\epsilon]\,]$

$\rightarrow N = \sum_r \frac{1}{e^{\beta(\epsilon_r-\mu)}-1}$

$\approx \int_{\epsilon_{min}}^{+\infty} \frac{g(\epsilon)d\epsilon}{e^{\beta(\epsilon_0-\mu)}-1}$ convergent if $\mu < \epsilon_{min}$

for $\mu \rightarrow \epsilon_{min} \Rightarrow$ BEC whether this happens $\sim g(\epsilon)$ in vicinity ϵ_{min}

Photons and black body radiation

model: cubic cavity of side L

inside photons in thermal equilibrium at temp T

→ Maxwell eq. $1/c^2\, \partial_t^2 \vec{E} = \nabla^2 \vec{E}$

solution: $\vec{E} = \vec{E}_0\, Re\{\exp(i(\vec{k}\cdot\vec{x} - \omega t))\}$ (traveling wave with speed c in direction $\vec{k}$)

→ dispersion relation: $\omega = c|\vec{k}|$

in a box ⇒ boundary conditions: assume conducting walls

$\vec{E} = \vec{E}_0 \sin(\vec{k}\cdot\vec{x})\sin(\omega t)$

with $\vec{k} = \frac{\pi}{L}(n_x, n_y, n_z)$ $n_{x,y,z} \in \mathbb{N}_0$, $(x,y,z) \in (0,L)$

then # of allowed waves in a volume element: $2V/\pi^3\, d\vec{k}$

photons have 2 transversal polarizations $(\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0)$

⇒ density of state $g(\omega)d\omega = \frac{1}{8}\ \frac{2V}{\pi^3}\ 4\pi k^2 dk = \frac{V}{\pi^2 c^3}\omega^2 d\omega$

number of osc. with freq $[\omega, \omega+d\omega]$ only $(n_x,n_y,n_z)>0$

Planck: treating photons as particles with energy $h\nu = \hbar\omega$

Planck: $\epsilon = \hbar\omega(n + 1/2)$ → ignore here

$\rightarrow \Xi = \sum_{\{n_\gamma\}} e^{-\beta\hbar\omega_\gamma n_\gamma} = \prod_\gamma \frac{1}{1-e^{-\beta\hbar\omega_\gamma}}$ → Bosons with $\mu = 0$

$\rightarrow \log \Xi = -\int_0^\infty d\omega\, g(\omega)\, \log(1 - e^{-\beta\hbar\omega})$

$V\omega^2/\pi^2 c^3$

$E = -\frac{\partial \log\Xi}{\partial\beta} = \int_0^\infty d\omega\, g(\omega)\, \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$ ↳ $x = \beta\hbar\omega$ (dimensionless)

$= \boxed{\frac{V\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^x-1}} = \frac{V\hbar}{\pi^2 c^3}\frac{1}{\beta^4\hbar^4}\underbrace{\int_0^\infty dx \frac{x^3}{e^x-1}}_{I}$

$= \frac{V\hbar}{c^3}\left(\frac{k_B T}{\hbar}\right)^4\left(\frac{I}{\pi^2}\right) = \frac{V\pi^2\hbar}{15c^3}\left(\frac{k_B T}{\hbar}\right)^4$

↳ Stefan-Boltzmann: $E \sim T^4$ $(E = \sigma T^4)$

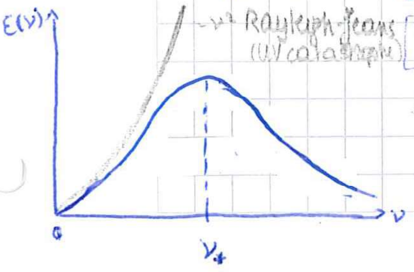
$I = \int_0^\infty dx\, x^3/e^x-1 = \int dx\, x^3 \sum_{n=1}^\infty e^{-nx} = \sum_{n=1}^\infty \int_0^\infty dx\, x^3 e^{-nx}$ ↪ 3 times integ. by parts

$= \sum_{n=1}^\infty \int_0^\infty \frac{6e^{-nx}}{n^3}dx = \sum_{n=1}^\infty \left(-\frac{6e^{-nx}}{n^4}\right)\Big|_0^\infty$

$= \sum_{n=1}^\infty \frac{6}{n^4} = 6\sum_{n=1}^\infty \frac{1}{n^4} = 6\,\zeta(4)$ $\pi^4/90$

$= \pi^4/15$

$E = \int d\omega\, \epsilon(\omega)$ ↳



$\sim \nu^2$ Rayleigh-Jeans (UV catastrophe)

$\epsilon(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$ Planck's law

→ energy dens per unit volume

expand

$\omega \gg 1 \rightarrow \epsilon \sim 0$

$\omega \ll 1 \rightarrow x \sim 0$: $\frac{x^3}{e^x-1} \approx x^2 - \frac{x^3}{2}$

$\Rightarrow \epsilon_p \underset{x\ll 1}{\approx} \frac{1}{\pi^2 c^3}\{\underbrace{\omega^2 k_B T}_{\text{RJ equip.}} - \underbrace{\frac{1}{2}\hbar\omega^3}_{\text{quantum corr.}} + \dots\}$

Fermions

non-interacting fermions:

$$\log \Xi = \sum_\gamma \log\left(1 + e^{-\beta(\varepsilon_\gamma - \mu)}\right)$$

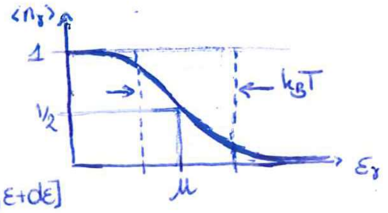
$$\rightarrow E = -\frac{\partial \log \Xi}{\partial \beta}\Big|_{\beta\mu} = \sum_\gamma \frac{\varepsilon_\gamma}{e^{\beta(\varepsilon_\gamma-\mu)}+1} = \sum_\gamma \langle n_\gamma\rangle \varepsilon_\gamma \approx \int_0^\infty \frac{\varepsilon g(\varepsilon)d\varepsilon}{1+e^{\beta(\varepsilon_\gamma-\mu)}}$$

$$\langle n_\gamma\rangle = \frac{1}{1+e^{\beta(\varepsilon_\gamma-\mu)}}$$

Fermi function

$g(\varepsilon)$ = dens. of states

$g(\varepsilon)d\varepsilon$ = # states in $[\varepsilon, \varepsilon+d\varepsilon]$



fermions → μ is not bounded, can take any possible value

- for $\varepsilon_0 = 0$: if $\mu < 0$ and $-\beta\mu \gg 1$ $\Rightarrow e^{-\beta\mu} \gg 1$

  $\rightarrow \langle n_\gamma\rangle \approx e^{-\beta(\varepsilon_\gamma-\mu)}$ Fermi → classical

  if $\mu > 0$ and $\beta\mu \gg 1$ $\Rightarrow$ distribution becomes sharper: ⅂

- for $T=0$: all states below Fermi energy ε_F occupied

  → degenerate Fermi system

  $$E_0 = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)d\varepsilon \quad \text{and} \quad N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon$$

=

I know it's double ...

define Fermi temperature $T_F = \varepsilon_F/k_B$: $T \ll T_F$ → $\langle n_\gamma\rangle$ is very sharp and almost fully degenerate

for $T \to 0$

$$\langle n_\gamma\rangle = \begin{cases} 1 & \varepsilon_\gamma < \varepsilon_F \\ 1/2 & \varepsilon_\gamma = \varepsilon_F \\ 0 & \text{else} \end{cases} \quad \text{with} \quad \varepsilon_F \equiv \lim_{T\to 0} \mu$$

thus $\lim_{T\to 0} E \equiv E_0 = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)d\varepsilon$

$$\lim_{N\to 0} N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon$$

$$\frac{PV}{k_BT} = \log \Xi \approx \int_0^\infty d\varepsilon\, g(\varepsilon) \log\left(1 + e^{-\beta(\varepsilon_\gamma-\mu)}\right)$$

$$\Longleftrightarrow \lim_{T\to 0} P = \lim_{T\to 0} \frac{1}{V}\int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)(\varepsilon_F - \varepsilon)$$

$$= \frac{1}{V}\int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)(\varepsilon_F-\varepsilon) = p_0 > 0$$

$$\lim_{T\to 0}\frac{1}{\beta}\log\left(1+e^{-\beta(\varepsilon_\gamma-\mu)}\right) = \begin{cases} -(\varepsilon_\gamma - \varepsilon_F) & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$$

for e^{large}: $\log(1+e^{\beta\dots}) \approx \log(e^{-\beta\dots}) = -\beta\dots$

→ pressure does NOT vanish

some fermions have non-zero momentum due to the exclusion principle

⇒ non-zero pressure

[in their ground state: occupy high mom. state → generate finite pressure]

low temperature behaviour

$$E = \int_0^{+\infty} d\varepsilon \frac{g(\varepsilon)\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu d\varepsilon\, g(\varepsilon)\frac{1+e^{\beta(\varepsilon-\mu)} - e^{\beta(\varepsilon-\mu)}}{1+e^{\beta(\varepsilon-\mu)}}\varepsilon + \int_\mu^{+\infty} d\varepsilon \frac{g(\varepsilon)\varepsilon}{1+e^{\beta(\varepsilon-\mu)}}$$

$$= \int_0^\mu d\varepsilon\, g(\varepsilon)\varepsilon - \int_0^\mu d\varepsilon \frac{g(\varepsilon)\varepsilon}{1+e^{-\beta(\varepsilon-\mu)}} + \int_\mu^{+\infty} d\varepsilon \frac{g(\varepsilon)\varepsilon}{1+e^{\beta(\varepsilon-\mu)}}$$

$$= \int_0^\mu d\varepsilon\, g(\varepsilon)\varepsilon - \frac{1}{\beta}\int_0^{\beta\mu} dx \frac{(\mu - x/\beta)g(\mu - x/\beta)}{1+e^x} + \frac{1}{\beta}\int_0^\infty dx \frac{(\mu + x/\beta)g(\mu + x/\beta)}{1+e^x}$$

$$= \int_0^\mu d\varepsilon\, g(\varepsilon)\varepsilon + \frac{1}{\beta}\int_0^{+\infty} dx \frac{(\mu + x/\beta)g(\mu+x/\beta) - (\mu - x/\beta)g(\mu - x/\beta)}{1+e^x}$$

sub with $x \equiv \mp\beta(\varepsilon - \mu)$

set $g(\varepsilon)=0$ for $\varepsilon<0$

assuming that $\varepsilon_0=0$

→ can extend 1st integr. without any problems

$$E = \int_0^{\mu} d\varepsilon\, g(\varepsilon)\varepsilon + k_BT\int_0^{+\infty} dx\, \frac{(\mu+k_BTx)g(\mu+k_BTx) - (\mu-k_BTx)g(\mu-k_BTx)}{1+e^x}$$

low temp T : expansion

$$g(\mu \pm k_BTx) \approx g(\mu) \pm k_BTx\, g'(\mu)$$

$$\Leftrightarrow (\mu \pm k_BTx)g(\mu \pm k_BTx) \approx \mu g(\mu) \pm k_BTx(\mu g'(\mu) + g(\mu))$$

(only keep smallest order term in k_BT)

$$\Rightarrow E \approx \int_0^{\mu} d\varepsilon\, g(\varepsilon)\,\varepsilon + 2(k_BT)^2 (\mu g'(\mu) + g(\mu)) \int_0^{\infty} \frac{x\,dx}{1+e^x}$$

$\int_0^{\infty} \frac{x dx}{1+e^x} = \frac{\pi^2}{12}$

as $T \to 0$
$\mu \to \varepsilon_F$

$$= \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)\varepsilon + \int_{\varepsilon_F}^{\mu} d\varepsilon\, g(\varepsilon)\,\varepsilon + \frac{\pi^2}{6}(\cancel{2}k_BT)^2 [\varepsilon_F g'(\varepsilon_F) + g(\varepsilon_F)]$$

$\int_x^{x+\Delta x} f(x)dx$
$= \int_x^{x+\Delta x} [f(x_0) + f'(x-x_0) + \dots]dx$
$= f(x_0)\Delta x + O(\Delta x^2)$

$$\approx \underbrace{\int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)\varepsilon}_{E_0} + g(\varepsilon)\varepsilon_F(\mu - \varepsilon_F) + (k_BT)^2[\varepsilon_F g'(\varepsilon_F) + g(\varepsilon_F)]\frac{\pi^2}{6}$$

similarly

$$N \approx \int_0^{+\infty} d\varepsilon\, \frac{g(\varepsilon)}{1+e^{\beta(\varepsilon-\mu)}} \approx \dots \approx \underbrace{\int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)}_{N_0} + g(\varepsilon_F)(\mu - \varepsilon_F) + \frac{\pi^2}{6}(k_BT)^2 g'(\varepsilon_F)$$

$$\Leftrightarrow (\mu - \varepsilon_F)g(\varepsilon_F) = -\frac{\pi^2}{6} g'(\varepsilon_F)(k_BT)^2 + O(N-N_0)$$

$$\Rightarrow E \approx E_0 - \varepsilon_F \frac{\pi^2}{6} g'(\varepsilon_F)(k_BT)^2 + \frac{\pi^2}{6}(k_BT)^2 \varepsilon_F g'(\varepsilon_F) + \frac{\pi^2}{6}(k_BT)^2 g(\varepsilon_F)$$

$$= E_0 + \frac{\pi^2}{6}(k_BT)^2 g(\varepsilon_F)$$

$$\Rightarrow C_V = +\frac{\partial E}{\partial T}\Big|_V \approx \frac{\pi^2}{3} g(\varepsilon_F) k_B^2 T$$ which vanishes for low T

in good approx: conducting e^- in metal ~ gas of free Fermions
@T_{room} : c_V contribution e^- masked by that of lattice vibrations ($\sim T^3$)
@T_{low} : $c_V \sim T$ e^- , no longer effect lattice vibr. (phonons)

$g(\varepsilon)$ for free particles:

$\vec{p} = h\vec{n}/L$

$\Rightarrow$ # states with mom. in $[p, p+dp]$

$$2\frac{V}{h^3}4\pi p^2 dp = \frac{16\pi V}{h^3} m^{3/2}\sqrt{2\varepsilon}\, d\varepsilon \equiv g(\varepsilon)d\varepsilon$$