Exam Differential Geometry

January 30, 2023

1. Consider the smooth function

$$F : \mathbb{R}^3 \to \mathbb{R} = F(x_1, x_2, x_3) = (x_1 + x_2)^2 - x_3^2.$$

   Let X be the vector field $X = x_3 \frac{\partial}{\partial x_2} + (x_1 + x_2)\frac{\partial}{\partial x_3}$.

   a) What are the regular values of F?

   b) For what regular values c is $X_p \in T_p(F^{-1}(c))$ for all $p \in F^{-1}(c)$?

2. a) Consider the map

$$G_t(x_1, y_1, x_2, y_2) = \begin{pmatrix} x_1 \cos t + x_2 \sin t \\ y_1 \cos t + y_2 \sin t \\ -x_1 \sin t + x_2 \cos t \\ -y_1 \sin t + y_2 \cos t \end{pmatrix}$$

   on $\mathbb{R}^4$ for $t \in R$. Prove or disprove: $\{G_t\}_{t\in\mathbb{R}}$ is a one-parameter group of diffeomorphisms.

   b) Let R be the vector field

$$R = y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2}$$

   and let F_s^R be its flow. Does

$$F_s^R \circ G_t = G_t \circ F_s^R$$

   for all s, t?

3. Let ω be the two-form

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4$$

on $\mathbb{R}^4$ and consider the function

$$G : \mathbb{R}^2 \to \mathbb{R}^4 : (x_1, x_2) \mapsto (x_1, x_2, g(x_1, x_2), h(x_1, x_2))$$

where $g, h : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

a) Compute the two-form $G^*\omega$ on $\mathbb{R}^2$ explicitly.

b) Prove or disprove: $G^*\omega = 0$ if and only of there exists a smooth function f on $\mathbb{R}^2$ such that $\frac{\partial f}{\partial x_1} = g$ and $\frac{\partial f}{\partial x_2} = h$.

4. Let G be a compact, connected Lie group of dimension n.

   a) Prove or disprove: if $f : G \to (\mathbb{R}, +)$ is a Lie group morphism, then f is constant.

   b) Let ω be a left-invariant n-form. Prove or disprove: ω is a right-invariant form.

      Hint: Consider $R_h^*\omega$ for $h \in G$.

5. Given an oriented manifold M of dimension m, state Stokes' Theorem.

6. Let E be a manifold of dimension $m + k$, and M be a manifold of dimension m. Let $\pi : E \to M$ be a rank-k vector bundle. Does there exists natural foliation of E of rank other than 0 or $m + k$?