Symmetries in Quantum Mechanics

Final Exam — Wednesday February 1, 2012

1. (a) What is the use of Clebsch-Gordan coefficients?

   (b) What is Kramers degeneracy?

   (Be brief.)

2. Symmetry under a parity transformation $\pi : (x, y, z) \to (-x, -y, -z)$ can give selection rules in addition to rotation symmetry selection rules. Explain why the same is not true for symmetry under a transformation $\pi' : (x, y, z) \to (-x, -y, z)$.

3. A system of two spineless particles with positions $\vec{r}_1$, $\vec{r}_2$ is described by the Hamiltonian $H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V$. For each of the following choices of interaction potential V, which of the observables (i) momentum, (ii) angular momentum, and (iii) parity are conserved?

   (a) $V = |\vec{r}_1|^4 + |\vec{r}_2|^4$

   (b) $V = |\vec{r}_1 - \vec{r}_2|^4$

   (c) $V = |\vec{r}_1 + \vec{r}_2|^4$

   (d) $V = |\vec{r}_1|^2 + |\vec{r}_2|^2 + \vec{r}_1 \cdot \vec{L}_1 + \vec{r}_2 \cdot \vec{L}_2$

   (e) $V = 1/|\vec{r}_1| + 1/|\vec{r}_2| + \vec{L}_1 \cdot \vec{L}_2$

   Consider both the total quantities and the quantities for the individual particles. Use geometric symmetry arguments rather than computations.

4. (a) A system of two up/down valued spins $\vec{S}_1$ and $\vec{S}_2$ is described by the Hamiltonian $H = \epsilon \vec{S}_1 \cdot \vec{S}_2$. Use $(\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$ and symmetry considerations to find a basis of energy eigenstates and the energy spectrum.

   (b) When the spins are placed in a magnetic field the Hamiltonian becomes $H = b(S_{1z} + S_{2z}) + \epsilon \vec{S}_1 \cdot \vec{S}_2$. Use the results you obtained in (4a) together with the fact that the new term is proportional to the z-component of the total spin to find the energy spectrum.

   (c) Find similarly the energy eigenvalues of a coupled triangle of three up/down valued spins $\vec{S}_i$ with Hamiltonian $H = \epsilon(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1) + \alpha(S_{1z} + S_{2z} + S_{3z})$.

5. Let $|n\ell m\rangle$ be energy eigenstates of a particle in a spherically symmetric potential. Use as many selection rules as you can to narrow down the values of ℓ and m for which the transition matrix element $\langle n\ell m|xyz|n20\rangle$ could be nonzero. (Hint: you may use the fact that the spherical harmonic $Y_3^{\pm 2} \propto (x \pm iy)^2 z$.)