GR Sessions 5: Gravitation

Wednesdays October 31, 2012

1. **Rotation.** Consider the 'rotating metric'

$$ds^2 = -\left(1 - \Omega^2\left(x^2 + y^2\right)\right)dt^2 + 2\Omega(y\,dx - x\,dy)dt + dx^2 + dy^2 + dz^2 \ ,$$

which represents a coordinate system spinning about the z-axis with angular velocity Ω in the clockwise direction. Prove that this metric is flat by way of a coordinate transformation.

2. **2d gravity.** Similar to the gauge invariance of electromagnetism $A_\mu \to A_\mu + \partial_\mu \lambda$, General relativity has a similar symmetry of the equations of motion called *diffeomorphism invariance*:

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ ,$$

where ξ_μ is an arbitrary vector.

   (a) Use this to argue that, in two spacetime dimensions, we can write $g_{\mu\nu} = e^\phi \eta_{\mu\nu}$.

   (b) What does this imply about the Riemann tensor (see §3.9 of Carroll), and why?

   (c) Use this to show that the Einstein tensor vanishes in two dimensions.

3. **Carroll 4.1** The Lagrange density for electromagnetism in curved space is

$$\mathcal{L} = \sqrt{-g}\left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) \ ,$$

where J^μ is the conserved current

   (a) Derive the energy momentum tensor by functional differentiation with respect to the metric. You can assume that the $A_\mu J^\mu$ term does not contribute to the energy-momentum tensor.

   (b) Consider adding a new term to the Lagrangian,

$$\mathcal{L}' = \beta\, R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \ .$$

   How are Maxwell's equations altered in the presence of this term? Einstein's equation? Is the current still conserved?

4. **Carroll 4.3** The four-dimensional δ-function on a manifold M is defined by

$$\int_M F\left(x^\mu\right)\left[\frac{\delta^{(4)}\left(x^\sigma - y^\sigma\right)}{\sqrt{-g}}\right]\sqrt{-g}d^4x = F\left(y^\sigma\right) \ ,$$

for an arbitrary function $F\left(x^\mu\right)$. Meanwhile, the energy-momentum tensor for a pressureless perfect fluid (dust) is

$$T^{\mu\nu} = \rho\, U^\mu U^\nu \ ,$$

where ρ is the energy density and U^μ is the four-velocity. Consider such a fluid that consists of a single particle travelling on a world line $x^\mu(\tau$, with τ the proper time. The energy momentum tensor for this fluid is then given by

$$T^{\mu\nu} = m\int_M \left[\frac{\delta^{(4)}\left(x^\sigma - y^\sigma\right)}{\sqrt{-g}}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau \ ,$$

where m is the rest mass of the particle. Show that covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, implies that $x^\mu(\tau)$ satisfies the geodesic equation.