GR Sessions 6: Einstein's Equations and Black Holes

Wednesdays November 14, 2012

1. **Escape from a black hole.** A spaceship hovers at coordinate point R outside of a Schwarzschild black hole. In order to escape from the black hole, the spacecraft must eject part of its rest mass, allowing for the remaining fraction f to reach infinity. Find the largest value of f that may escape to infinity as a function of R. What happens to f as $R \to 2GM$.

2. **Carroll 5.4** Consider Einstein's equations in vacuum, but with a cosmological constant, $G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0$.

   (a) Solve for the most general spherically symmetric metric, in coordinates (t, r) that reduce to the ordinary Schwarzchild coordinates when $\Lambda = 0$.

   (b) Write down the equation of motion for radial geodesics in terms of an effective potential. Sketch the effective potential for massive particles.

3. **Carroll 6.4** Consider de Sitter space in static coordinates:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2\, d\Omega^2 \ .$$

   This space has a Killing vector ∂_t that is timelike near $r = 0$ and null on a Killing horizon. Locate the radial position of the Killing horizon, r_K. What is the surface gravity, κ, of the horizon? Consider the Euclidean signature version of de Sitter space obtained by making the replacement $t \to i\tau$. Show that a coordinate transformation can be made to make the Euclidean metric regular at the horizon, so long as τ is made periodic.

4. **Carroll 5.5** Consider a comoving observer sitting at constant spatial coordinates $(r_\star, \theta_\star, \phi_\star)$, around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\rm em}$ (in the beacon rest frame).

   (a) Calculate the coordinate speed dr/dt of the beacon, as a function of r.

   (b) Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r, with a locally intertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?

   (c) Calculate the wavelength $\lambda_{\rm obs}$ measured by the observer at $r_\star$ as a function of the radius $r_{\rm em}$ at which the radiation was emitted.

   (d) Calculate the time $t_{\rm obs}$ at which a beam emitted by the beacon at radius $r_{\rm em}$ will be observed at $r_\star$.

   (e) Show that at late times, the redshift grows exponentially: $\lambda_{\rm obs}/\lambda_{\rm em} \propto e^{t_{\rm obs}/T}$. Give an expression for the time constant T in terms of the black hole mass M.